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Abstract

During the financial crisis, financial firm leverage and volatility both rose dramatically. Consequently, institutions are being asked to reduce leverage in order to reduce risk, though the effectiveness depends upon the role of capital structure in volatility. To address this question, we build a statistical model of equity volatility that accounts for leverage. Our approach blends Merton's insights on capital structure with traditional time-series models of volatility. Using our model we quantify how capital injections impact the risk of financial institutions and estimate firm specific precautionary capital needs. In addition, the longstanding observation that volatility is more responsive to negative shocks than positive is shown to be less a consequence of actual leverage than it is of risk premiums.

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1   Introduction

The financial crisis revealed the damaging role of financial market leverage on the real economy. Nonetheless, it is far from clear that reducing this leverage will stabilize the real economy, let alone stabilize the financial sector. The extreme volatility of asset prices was a joint consequence of the high impact of economic news and high leverage. A critical question that remains is how much reduction in equity volatility could be expected from reductions in leverage. The answer to this question will imply an estimate of the capital that a financial firm should have in order to achieve certain stability targets.

To address this issue, we introduce a statistical model of equity volatility that accounts for capital structure. We do so by incorporating the insight from Merton (1974) into a GARCH volatility model following Engle (1982) and Bollerslev (1986). In addition, our model nests models of volatility asymmetry (Nelson (1991); Glosten, Jagannathan, and Runkle (1993)) that have been widely discussed in the intervening years. Given its theoretical underpinnings, we call this new model Structural GARCH.

The key feature of our Structural GARCH model is that the risk of future equity returns depends on a firm's current capital structure. This observation dates back to at least Modigliani and Miller (1959), but has yet to make its way into the vast literature on volatility modeling. In our model, unobserved asset returns follow a GARCH process, and are amplified by what we call a leverage multiplier to generate equity returns. The leverage multiplier is derived from Merton's observation that equity is a call option on the underlying assets of the firm with a strike price that is the face value of the liabilities and a maturity given by the due date of the liabilities. Under reasonable assumptions, the leverage multiplier becomes a simple function of observable leverage, and its exact functional form depends on easily estimated parameters.

We estimate the Structural GARCH model for a sample of the one hundred largest U.S. financial firms. Our empirical results show that incorporating leverage into the GARCH framework is very useful for capturing the dynamics of financial firm equity volatility. The Structural GARCH model outperforms a standard GARCH model in a few ways. First, the Structural GARCH is favored in a statistical sense by a majority of the firms in our sample. This follows from the fact that our model nests the GARCH family, and thus provides a straightforward way to assess the statistical significance of leverage for equity volatility. Second, we compare the ability of the Structural GARCH model

to forecast high-frequency realized variance. Compared to the standard GARCH, our Structural GARCH model is superior for most firms in our sample as well.

Our model delivers estimates of time-varying equity volatility and asset volatility, as well as the leverage multiplier that links the two. Using these outputs, we start by decomposing the massive rise in financial sector equity volatility during the financial crisis, which reached almost 200% in annualized terms. To put this in perspective, the VIX index reached about 60% during this same time period. During the early phase of the crisis (May 2007 – Sept 2008), the rise in equity volatility was driven in large part by a steady increase in the leverage multiplier of the financial sector. At its peak, the financial sector leverage multiplier hits 20, indicating that asset shocks are amplified by a factor of 20. In contrast, the rise in financial sector asset volatility did not really occur until late in 2008.

Next, we use the Structural GARCH model to quantify how capital injections impact the risk of financial institutions. As a demonstrating example, we put ourselves in the position of U.S. regulators standing in October 2008 and evaluate the potential impact of a $25 billion equity injection into Bank of America.[1] Unsurprisingly, at all horizons, adding more capital lowers the volatility of equity returns and reduces the likelihood of extreme events. According to our model, injecting Bank of America with capital lowers the probability of going bankrupt over the next month by a factor of nearly three. It is of course up to the regulator to assess the importance of this effect, and in turn, how large of a capital injection to require — our model presents a way to quantify this tradeoff.

Building on this analysis, we then define a new measure of systemic risk for financial institutions, and we refer to this measure as precautionary capital. Precautionary capital is the answer to the question: how much equity do we have to add to a firm today in order to ensure some arbitrary level of confidence that the firm will have adequate capital in a future crisis? The issue of precautionary capital is thus highly related to the Federal Reserve's stress testing (CCAR) of U.S. financial institutions. In the Structural GARCH model, we quantitatively show that preventative measures that reduce leverage can be very powerful since holding more capital results in lower volatility, lower beta, and lower probability of failure. This is a sensible outcome that is not implied by conventional time-series volatility models because capital structure has no

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[1] This corresponds to the size of the first equity injection BAC would actually receive from the U.S. government.

explicit role in determining risk dynamics.

Because our model links leverage and volatility, it also sheds light on the sources of volatility asymmetry. Volatility asymmetry describes situations where negative equity returns predict higher future volatility, relative to positive equity returns of the same magnitude. The original explanation for this phenomenon dates back to Black (1976) and Christie (1982), who ascribe it to a mechanical leverage effect: declines in equity today mechanically increase leverage, and because higher leverage is associated with more risk, future equity volatility also rises. French, Schwert, and Stambaugh (1987) observe that risk premiums could also account for volatility asymmetry. In their alternative, a rise in future volatility raises the required return on equity, leading to an immediate decline in the stock price.

One challenge in distinguishing between these two explanations is that asset returns are unobservable, so teasing out the causes of volatility asymmetry typically requires alternative strategies (Duffee (1995); Bekaert and Wu (2000); Choi and Richardson (2015)). The Structural GARCH model provides a simple way to estimate asset returns and volatility, while crucially allowing for the debt of each firm to be risky. We find that almost all of the volatility asymmetry that is present in equity returns is still present in asset returns. This cuts strongly against the mechanical leverage effect explanation. We then compute idiosyncratic asset returns by purging asset returns of exposure to priced risk factors (Fama and French (1993)). In our sample, idiosyncratic asset returns display very little volatility asymmetry, which lends support to the risk premium explanation of asymmetry.

Literature Review

At its core, the Structural GARCH is a time-series model of volatility that explicitly allows leverage to impact equity volatility. As the name suggests, the way in which leverage interacts with equity volatility is motivated by structural models of credit (Merton (1974)). A non-exhaustive list of theoretical extensions of the Merton (1974) model includes Black and Cox (1976), Leland and Toft (1996), and Collin-Dufresne and Goldstein (2001). Because our model allows asset volatility to be time-varying, we also connect to variants of the Merton (1974) model with stochastic volatility (e.g. McQuade (2013), Elkamhi, Ericsson, and Jiang (2011), and Fouque, Sircar, and Sølna (2006)).

The exact stochastic nature of asset returns in our model draws from time-series

models of volatility (Engle (1982) and Bollerslev (1986)). Nelson (1991), Glosten, Jagannathan, and Runkle (1993), and Engle and Ng (1993) are notable extensions of the ARCH/GARCH class that indirectly allow capital structure to impact future volatility. They do so by allowing negative returns to impact future volatility differently than positive returns. The basic idea behind these models is that negative shocks to equity raise leverage and therefore may raise equity risk more than positive shocks of the same magnitude (so-called "volatility asymmetry"). The Structural GARCH is closely related to this set of models, but makes an important departure in allowing capital structure to directly impact volatility dynamics. Intuitively, in our model, the level of leverage determines the level of equity volatility, while still allowing for volatility asymmetry in asset returns.[2]

This approach allows us to disentangle the causes of volatility asymmetry, which as mentioned dates back to Black (1976), Christie (1982), and French, Schwert, and Stambaugh (1987). Our main finding is that mechanical leverage drives almost none of the observed equity volatility asymmetry for the firms in our sample. These results are consistent with, among others, Bekaert and Wu (2000) and Hasanhodzic and Lo (2013). Both studies find that leverage does not appear to fully explain the asymmetry in equity volatility. In contrast to the current paper, Bekaert and Wu (2000) assume that debt is riskless and are therefore silent about the nonlinear interaction between equity volatility and leverage. Hasanhodzic and Lo (2013) focus on a subset of firms with no leverage, which we do not pursue in this paper. Choi and Richardson (2015) also study volatility asymmetry by invoking the second Modigliani–Miller theorem. In turn, they directly compute the market value of assets at a monthly frequency by first constructing a return series for the market value of bonds (and loans). Still, determining the true market value of the bonds of a given firm is difficult in lieu of liquidity issues, especially at the daily frequency with which our model operates.

Our paper also connects to a rapidly growing systemic risk literature that emphasizes the consequences of undercapitalization of financial institutions. Acharya, Pedersen, Philippon, and Richardson (Forthcoming) develop a model where the social cost of undercapitalized banks is greater than the private cost. Consequently, there are incentives to take more leverage and risk than is socially optimal. Acharya, Engle, and

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[2]Carr and Wu (Forthcoming) use this intuition in building a reduced-form model of leverage and equity variance to study the pricing of S&P 500 index options. Geske, Subrahmanyam, and Zhou (2016) also investigate the impact of leverage on option prices using a compound option pricing model.

Richardson (2012) and Brownlees and Engle (Forthcoming) develop an econometric approach to measuring the capital shortfall based on comovements of financial stocks and the broad market. Other popular measures of fragility are based solely on equity data such as the CoVaR model of Adrian and Brunnermeier (2016) and the network models of Billio, Getmansky, Lo, and Pelizzon (2012) and Diebold and Yilmaz (2014).

These academic studies are closely linked to the new developments in macroprudential regulation since Dodd Frank and Basel II and III. Following these protocols, bank stress tests are carried out in order to measure whether banks are sufficiently well capitalized. There are two risks to be considered when the banking sector as a whole is undercapitalized. The first is that there will be an exogenous shock that will be sufficiently large that highly leveraged financial firms will fail and cause a severe decline in output. The second is that the undercapitalization itself will bring about the decline. As the entire banking sector tries to reduce its risk by deleveraging, it will drive down the value of assets that are being sold and create a downward spiral of bank valuations through the fire sale externality. This has been discussed by Brunnermeier and Pedersen (2009), among others. Typically in stress tests, the regulator will estimate the capital adequacy ratio under stress and give either a pass or a fail to the bank which may then have to revise its capital plans. Neither the regulators nor the banks actually estimate the capital shortfall. Presumably this is because it is difficult to estimate the impact of changes in capital structure on volatility and on correlation. Thus the model developed in this paper could assist both regulators and banks in estimating the extent of capital augmentation needed to pass stress tests.

The remainder of the paper is organized as follows. Section 2 develops our workhorse econometric model. Section 3 provides details on how we take our model to the data and analyzes the resulting estimates for our cross section of financial firms. In this section, we also consider some additional model validation to further emphasize the importance of accounting for the connection between equity volatility and leverage. Section 4 presents two applications of our Structural GARCH model: (i) measuring systemic risk and (ii) understanding the cause of volatility asymmetry in equity returns. Section 5 concludes the paper by suggesting additional uses of our new model.

2 The Model

2.1 Motivation

The ARCH/GARCH Framework

Our simple goal is to incorporate capital structure into the ARCH/GARCH class of volatility models. These models typically take the following form:

$$r_{E,t} = \sigma_{E,t} \times \varepsilon_{E,t}$$

where $r_{E,t}$ denotes demeaned equity returns and $\varepsilon_{E,t}$ is a mean zero and variance one shock. $\sigma_{E,t}$ is the conditional volatility of equity returns, and in the standard GARCH model, it is conditioned on past information information ($\mathcal{F}_{t-1}$) as follows:

$$
\begin{aligned}
\sigma_{E,t}^2 &= \mathbb{E}\left[r_{E,t}^2|\mathcal{F}_{t-1}\right] \\
&= \mathbb{E}\left[r_{E,t}^2|r_{E,t-1}, r_{E,t-2}, ...r_{E,1}\right]
\end{aligned}
$$

This conditional expectation is typically parametrized as a recursive process $\sigma_{E,t}^2 = \omega + \alpha r_{E,t-1}^2 + \beta \sigma_{E,t-1}^2$, though more elaborate recursive structures can be handled easily. In this paper, we add leverage to the set of conditioning variables:

$$\sigma_{E,t}^2 = \mathbb{E}\left[r_{E,t}^2|Leverage_{t-1}, r_{E,t-1}, r_{E,t-2}, ...r_{E,1}\right] \tag{1}$$

The basic logic of our approach is to use structural models of credit to provide economic discipline in terms of how we introduce leverage into Equation (1).

Structural Models of Credit

To motivate the economic relationship between leverage and equity volatility, we start from the observation that a firm's equity can be viewed as a call option on the underlying assets of the firm (Merton (1974)):

$$E_t = f\left(A_t, D_t, \sigma_{A,t}, \tau_t, r_t; \Theta_p, \Theta_r\right) \tag{2}$$

where $f(\cdot)$ is an unspecified call option function, A_t is the current (unobservable) market value of assets and D_t is the current book value of outstanding debt, which we in-

terpret as the face value of debt. $\sigma_{A,t}$ is the potentially stochastic volatility of the assets, τ_t is the life of the debt in years, and r_t is the annualized risk-free rate at time t. Θ_p is a vector of parameters that govern the asset dynamics under the physical $\mathbb{P}$-measure. Θ_r is a vector of parameters that describe the pricing of risks and derives from the underlying preference parameters that enter the stochastic discount factor in the economy.

In Appendix A.1 we argue that Equation (2) implies that equity volatility (under the $\mathbb{P}$-measure) can be well-approximated as:

$$\sigma_{E,t} = LM\left(D_t/E_t, \sigma_{A,t}, \tau_t, r_t; \Theta_p, \Theta_r\right) \times \sigma_{A,t} \tag{3}$$

where the function $LM(\cdot)$ is what we call the "leverage multiplier". To derive this relationship, we assume a generic process for asset returns that nests many popular option pricing settings from the previous literature.[3] We then compute the law of motion for equity returns using repeated applications of Ito's Lemma. Finally, we show that for reasonable parameter values governing asset return dynamics, equity volatility is a scaled function of asset volatility, where the function depends on financial leverage. Thus, LM captures how financial leverage amplifies asset volatility to generate equity volatility. Intuitively, the exact functional form of the leverage multiplier is ultimately dictated by the specific call option pricing function $f(\cdot)$ from Equation (2); however, as we show shortly, it has some basic properties that do not appear to depend on the particular option pricing setting. When it is obvious, we will drop the functional dependence of the leverage multiplier on its arguments and instead denote it simply by LM_t.

Some Examples and Intuition

To develop some intuition of how the leverage multiplier depends on financial leverage, Figure 1 plots the leverage multiplier under some different option pricing models. In particular, we consider three different settings: (i) the Black-Scholes-Merton (BSM) model; (ii) the Bakshi, Cao, and Chen (1997) model (BCC), which is an extension of the BSM model that allows for stochastic volatility and jumps; and (iii) a discrete time model where risk-neutral asset returns evolve according to an asymmetric GARCH process (Glosten, Jagannathan, and Runkle (1993), Barone-Adesi, Engle, and

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[3]A short and certainly incomplete list includes Black and Scholes (1973), Heston (1993), Stein and Stein (1991), and Bates (1996).

Mancini (2008), Duan (1995)).[4] Studying the leverage multiplier in these environments is useful because they span a variety of assumptions regarding discrete and continuous time, stochastic volatility, and jumps. Appendix B contains additional details on how we parametrize and compute the leverage multiplier in these different settings. In all cases, we are measuring leverage as the book value of debt divided by the market value of equity.

Figure 1 reveals interesting differences in the leverage multiplier across the three models. For instance, the leverage multiplier in the BCC model is larger than the BSM multiplier for all values of leverage. This occurs because the BCC asset return process has jumps that are negative on average, and also because there is a negative correlation between asset volatility shocks and the Gaussian asset return shocks. The combination of both forces means that the asset return distribution in the BCC model is more left skewed compared to the BSM model. In turn, leverage leads to more amplified equity volatility because high leverage corresponds to a much smaller likelihood the equity expires "in the money." A similar intuition holds when looking at the leverage multiplier when risk-neutral asset returns follow an asymmetric GARCH process. The way we have parameterized the asymmetric GARCH process in Figure 1 implies that volatility innovations and asset return innovations are negatively correlated, thereby leading to a large negative skew in asset returns (Engle (2011)).

On the other hand, Figure 1 also highlights that the leverage multiplier has some characteristics that are common across the different option pricing models. For one, when leverage is zero, the leverage multiplier is one. This result is mechanical given that zero leverage means that the market value of assets is exactly the market value of equity, so their volatilities have to coincide. A second common feature is that the leverage multiplier is monotonically increasing in leverage. Intuitively, as a firm becomes more leveraged, equity volatility increases for a given level of asset volatility. Finally, in all three models, the leverage multiplier appears to be concave in leverage. The intuition for this result is less clear, but one way to rationalize the concavity of the leverage multiplier is to consider a firm with extremely high leverage. In this case, the equity of the firm is close to worthless, so adding an additional unit of leverage does not add much more in terms of volatility amplification.[5] Motivated by the fact that some char-

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[4]McQuade (2013), Elkamhi, Ericsson, and Jiang (2011), and Fouque, Sircar, and Sølna (2006) are a few additional examples of efforts made in the credit risk literature to extend the Merton model to account for stochastic volatility.

[5]We believe that all three of these properties are general features that likely derive from no arbitrage

acteristics of the leverage multiplier appear independent of a specific option pricing model, we now propose a function to capture the leverage amplification mechanism in a relatively "model-free" way.

2.2   A Flexible Leverage Multiplier

One approach to quantifying how leverage amplifies asset volatility into equity volatility would be to choose a particular option pricing model, fit the parameters of that model to observed leverage levels, and then use the estimated parameters to trace out the leverage multiplier. The downside to this approach is that it forces one to take a stance on an option pricing model. However, an alternative route that we take is to model the leverage multiplier directly, without imposing a specific option pricing model on the data. Specifically, we propose the following function of leverage:

$$ LM\left(\frac{D_t}{E_t}, \sigma_{A,t}, \tau_t, r_t; \phi\right) = \left[LM^{BSM}\left(\frac{D_t}{E_t}, \sigma_{A,t}, \tau_t, r_t\right)\right]^{\phi}, \qquad \phi \geq 0 \qquad (4) $$

where LM^{BSM} is the leverage multiplier in the Black-Scholes-Merton model. We provide a more explicit expression for LM^{BSM} in Appendix A.2, as well as additional details on how to compute it using observed values of leverage.

It is critical to recognize that we are *not* assuming that the underlying option pricing model is the Black-Scholes-Merton model, and as such, the success of our empirical approach does not rest on the ability of the BSM model as a structural model of credit. Instead, our leverage multiplier simply uses a mathematical transformation of the BSM functions. This is captured by the new parameter ϕ, which will be an estimated parameter when we take our model to the data. One advantage of our proposed leverage multiplier in Equation (4) is its relative simplicity in terms of computation, as the BSM leverage multiplier is numerically tractable. This specification is also a parsimonious way to capture some of the features of the leverage multiplier documented in Section 2.1, while still maintaining some level of agnosticism over the true option pricing model that drives volatility amplification.[6]

arguments, similar to those put forth in Merton (1973). While it is straightforward to show this in a given option pricing model, we have been unsuccessful in proving this formally.

[6]To be precise, ϕ preserves the concavity of the leverage multiplier with respect to leverage as long as it is not too large, long-run asset volatility is not too small, and τ is not too small. In practice, this is not an issue. When we estimate the model, we verify that none of the fitted ϕ result in violations of this sort.

Figure 2 demonstrates the flexibility of our proposed leverage multiplier. In the figure, we have reproduced the leverage multipliers from the Bakshi, Cao, and Chen (1997) and the GARCH models that we explored in the previous subsection. These correspond to the solid lines with square and triangle markers, respectively. In addition, we have estimated two different values of ϕ in our proposed leverage multiplier to best match the two models. The leverage multipliers from this exercise are plotted in dotted lines, and the legend in the figure displays the estimated value of ϕ in both cases. The simple takeaway from Figure 2 is that toggling ϕ in our proposed function is effective in replicating the leverage multiplier in both the BCC and GARCH settings, despite the fact that they have very different features (e.g. jumps, continuous versus discrete time, etc.).

2.3   The Full Recursive Model

The preceding analysis motivates the use of our leverage multiplier in describing the relationship between equity volatility and leverage. To make the model fully operational in discrete time, we propose the following recursive process for equity returns:

$$
\begin{aligned}
r_{E,t} &= LM_{t-1}r_{A,t} \\
r_{A,t} &= \sqrt{h_{A,t}}\varepsilon_{A,t}, \quad \varepsilon_{A,t} \sim D(0,1) \\
h_{A,t} &= \omega + \alpha\left(\frac{r_{E,t-1}}{LM_{t-2}}\right)^2 + \gamma\left(\frac{r_{E,t-1}}{LM_{t-2}}\right)^2 1_{r_{E,t-1}<0} + \beta h_{A,t-1} \\
LM_{t-1} &= \left[LM^{BSM}\left(\frac{D_{t-1}}{E_{t-1}}, \sigma_{A,t-1}^{\tau}, \tau_{t-1}, r_{t-1}\right)\right]^{\phi}
\end{aligned}
\tag{5}
$$

where $\varepsilon_{A,t}$ has a mean zero and variance one. We call the specification described in Equation (5) a "Structural GARCH" model. To see how this model maps to the motivation in the previous sections, notice that equity variance $h_{E,t}$ is given by:

$$
h_{E,t} = LM_{t-1}^2 h_{A,t}
$$

which is exactly the relationship that is implied by structural models of credit (see Equation (3)). Here, $h_{A,t}$ represents the variance of asset returns, $r_{A,t}$. It evolves according to the volatility model introduced by Glosten, Jagannathan, and Runkle (1993) (GJR), which is a member of the GARCH family. This is easiest to see by observing that

$r_{E,t-1}/LM_{t-2} = r_{A,t-1}$, which means that asset variance can be expressed as the more standard GJR recursion:

$$h_{A,t} = \omega + \alpha r_{A,t-1}^2 + \gamma r_{A,t-1}^2 1_{r_{A,t-1}} + \beta h_{A,t-1}$$

We chose the GJR model for asset variance because it nests a standard GARCH model ($\gamma = 0$) and has been shown to display superior performance when applied to equity returns directly (Engle and Ng (1993)). The GJR formulation for asset variance will also prove useful when we use the Structural GARCH model to examine volatility asymmetry in Section 4.2, though a number of alternative volatility models (e.g. Nelson (1991)) would serve this purpose. The total parameter set for the Structural GARCH is $\Theta := (\omega, \alpha, \gamma, \beta, \phi)$. Importantly, there is only one extra parameter compared to a simple GJR model, though one could add additional lags and parameters in a natural way. Also note that we have formulated the Structural GARCH model for an arbitrary firm, but in our empirical work we will estimate a separate model for each firm in our sample.

With some abuse of notation, we use $\sigma_{A,t-1}^\tau$ to denote the expected cumulative asset volatility over the life of the debt. This is different than (but highly related to) the instantaneous volatility of assets $h_{A,t}$. We will confront the issue of how to compute τ_t and how to use the volatility input $\sigma_{A,t-1}^\tau$ in the next section when describing the data and estimation techniques used in our empirical work. We also introduce lags in the appropriate variables (e.g., the leverage multiplier) to ensure that one-step ahead volatility forecasts are indeed in the previous day's information set. An attractive feature of the model in (5) is that it nests a simple GJR model ($\phi = 0$), thus providing a statistical test of how leverage affects equity volatility.

Additional Discussion

A Feedback Mechanism

Another advantage of our equity volatility model is that it explicitly allows for feedback between leverage and volatility. For example, when simulating this model, if a series of negative asset returns is realized (and hence negative equity returns since they share the same shock), volatility rises more due to the asymmetric specification inherent in the GJR. Negative equity returns also lead to an increase in leverage, which in turn increases the leverage multiplier and results in an even stronger amplification effect for

equity volatility. As we saw in the recent financial crisis, this was a key feature of the data, particularly for highly leveraged financial firms.

Asset Levels versus Asset Returns

The specification in Equation (5) also suggests that we can recover asset returns as the ratio of equity returns to the lagged leverage multiplier. However, it is important to recognize that we are not able to recover the level of the market value of assets. This is because we have not taken a stand on a particular option pricing model that is driving the data. Instead, our estimated leverage multiplier tells us how shocks to asset returns get scaled into equity returns.

Physical Versus Risk-Neutral Volatility

A subtle and perhaps surprising aspect of our model is that it describes asset returns and volatility under the physical measure, as opposed to the risk neutral measure. At first glance, this may seem puzzling given that the observed equity values are options on the underlying assets, so one might think we can only say something about the risk-neutral asset return process. To illustrate why we are able to learn something about asset returns under the physical measure, suppose the true model of the world was the Heston (1993) stochastic volatility option pricing model. In this case, asset returns under the physical measure evolve according to:

$$
\begin{aligned}
\frac{dA_t}{A_t} &= \mu_A dt + \sigma_{A,t} dB_A(t) \\
d\sigma^2_{A,t} &= \kappa \left[\theta - \sigma^2_{A,t} \right] dt + \sigma_v \sigma_{A,t} dB_v(t), \quad corr(dB_A(t), dB_v(t) = \rho dt
\end{aligned}
$$

Using the notation from our motivating setup in Section 2.1, this means that $\Theta_p = (\mu_A, \kappa, \theta, \sigma_v, \rho)$. Additionally, in the Heston (1993) model, the pricing of volatility risk is dictated by an additional parameter, $\Theta_r = \lambda$. In turn, the leverage multiplier in the Heston (1993) model also depends on λ. Formally, this means that equity under the $\mathbb{P}$-measure in the Heston model evolves (approximately) as:

$$
\frac{dE_t}{E_t} = LM_t^{Heston} \left(E_t, D_t, \sigma_{A,t}, r_{ft}, \tau_t; \Theta_p, \lambda \right) \times \frac{dA_t}{A_t}
$$

where LM_t^{Heston} is naturally the leverage multiplier implied by the Heston (1993) call

option function. Thus, when translating between equity volatility and asset volatility under the $\mathbb{P}$-measure as in Equation (3), the leverage multiplier contains all of the risk adjustment λ.

In practice, we approximate the true leverage multiplier and asset returns with our proposed specification in Equation (4). In other words, our working assumption is that:

$$LM_t^{Heston}\left(E_t, D_t, \sigma_{A,t}, r_{ft}, \tau_t; \Theta_p, \lambda\right) \approx \left[LM^{BSM}\left(\frac{D_t}{E_t}, \sigma_{A,t}^\tau, \tau_t, r_t\right)\right]^\phi$$

for some value of ϕ. This makes clear that ϕ is absorbing any risk adjustment terms in Θ_r. As our analysis in Section 2.2 shows, this appears to be a reasonable assumption. To see this most clearly, see the derivation of Equation (3) that appears in Appendix A.1. The larger implication of this discussion is that our model provides estimates of asset dynamics under the $\mathbb{P}$-measure.

To provide some more support for this argument, we conduct a simple calibration exercise. Specifically, we compute the leverage multiplier in the Heston (1993) model using the following parameter values: $\kappa = 4.16$, $\theta = 0.0204$, $\sigma_v = 0.175$, $\rho = -0.7$. These parameters correspond to a half-life for asset volatility shocks of 2 months, long-run asset volatility of 14.3%, and a volatility of asset volatility of 17.5%. All of these are parameters under the physical measure. We vary λ from -1.39, -2.77, and -3.23; this means that volatility has a negative price of risk, which is equivalent to volatility being more persistent and having a larger long-run average under the risk-neutral measure, relative to the physical measure. We chose these specific values of λ such that the half-life of risk-neutral volatility would range from 3-9 months (Christoffersen, Heston, and Jacobs (2009)).[7] We use a risk-free rate of $r = 0.03$ and time to maturity $\tau = 2$.

Figure 3 plots the leverage multiplier from the Heston (1993) model for these parameter combinations. In addition, we use the long-run average of physical volatility (14.3%) as an input into our leverage multiplier, and then compute ϕ to best match each parameter combination. The solid lines in the figure show the true leverage multiplier from the Heston (1993) model, and the dotted lines show our fitted leverage multiplier for each case. As is clear from these plots, our leverage multiplier is a good approximation because ϕ effectively absorbs the risk-adjustment contained in λ.

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[7]Equivalently, for $\lambda = \{-1.39, -2.77, -3.23\}$, this means risk-neutral volatility has a long-run mean of 17.5%, 24.7%, and 30.3% respectively.

3 Estimation Details and Results

3.1 Data Description and Estimation Considerations

We now turn to estimating the our volatility model using equity return data. We estimate the model using standard quasi-maximum likelihood techniques (Bollerslev and Wooldridge (1992)). Unless otherwise noted, we obtain all of our data from Datastream. The set of firms we analyze are 91 financial firms over a period that starts in 1998 and ends in June 2016.[8] The exact date span obviously varies from firm to firm. A full description of the set of firms is contained in Appendix F. We choose this set of firms based on the total book value of assets according to Datastream as of 7/15/2016. The reasons we focus on financial firms are twofold: first, these firms typically have fairly high leverage and so we would expect our leverage-volatility model to apply best to these types of firms. Second, given the high volatility in the recent crisis that was accompanied by unprecedented leverage, this set of firms presents an important sector to model from a systemic risk and policy perspective. To this end, one of the applications of our model that we will explore in later sections involves systemic risk measurement of financials.

The Face Value of Debt

We define the face value of debt D_t as the sum of insurance reserves, deposits, short term debt, long term debt, and other liabilities.[9] Datastream reports debt values at the end of each quarter; for each day in a given quarter, we use the debt value reported at the end of the last quarter. This naturally creates a very choppy debt series. To minimize the impact that this has on the estimation, we smooth the daily book value of debt using an exponential average:

$$D_t = \eta \widehat{D}_t + (1 - \eta) D_{t-1}, \qquad D_0 = \widehat{D}_0$$

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[8]Note that we drop 9 firms from our original sample of 100 firms. Two of the firms, FNMA and FMCC, both had average debt-to-equity ratios of over 1000, so we excluded them to avoid undue influence of outliers. We exclude the remaining 7 firms because they had unusual parameters when we estimated a simple GJR model to their equity returns. The parameter β in the variance recursion $h_t = \omega + \alpha r_{t-1}^2 + \gamma r_{t-1}^2 1_{r_{t-1}<0} + \beta h_{t-1}$ is typically greater than 0.8, which reflects the highly persistent nature of volatility. For these 7 firms, the estimated β was well below 0.7, likely driven by the fact that these firms had some missing returns and/or short time series.

[9]Insurance reserves is the actuarial present value of all insurance policies for the firm. Unsurprisingly, this liability is relevant mainly for the insurance companies in our sample.

where D_t is the smoothed debt series that we feed into the model and $\widehat{D}_t$ is the actual observed debt series. We set the smoothing parameter to $\eta = 0.01$, which implies a half-life of approximately 70 days in terms of the weights of the exponential average. This seems reasonable for quarterly data. In addition, smoothing the debt in this way ensures that the D_t series is computed using information only up to time t. In the Online Appendix, we also present some robustness analysis showing that our results are not impacted by the choice of the smoothing parameter.

The Risk Free Rate

To compute the leverage multiplier, we must also input the risk free rate over the life of the debt. We do so by using a zero-curve provided by OptionsMetrics, which is derived from BBA LIBOR rates and settlement prices of CME Eurodollar futures. The zero-curve data from OptionsMetrics reports data for a limited number of maturities, so we build the term structure for all maturities by simple linear interpolation. Finally, we assume a flat term structure for all tenors beyond the maximum maturity reported by OptionsMetrics.

Debt Maturity

To compute the debt maturity for every firm at each point in time, we first assign a maturity to each type of liability on the firm's balance sheets. For all firms, we set the maturity of each liability as follows: insurance reserves are 30 years, deposits are 1 year, short term debt is 2 years, long term debt is 8 years, and other liabilities are 3 years. These maturities reflect our best guess of the "typical" maturity of each type of liability. For example, the maturity of insurance reserves should reflect the average maturity of insurance contracts that have been promised by a given institution; hence, a maturity of 30 years is sensible because the insurance companies in our sample (e.g. MetLife) are in large part exposed to life insurance contracts that have long coverage periods.

For each firm, we then take a weighted average of these maturities, where the weights are given by the proportion of total debt that is attributable to each liability. To provide a concrete example, consider the calculation of Bank of America's (BAC) debt maturity on 7/15/2016. On this date, the proportion of BAC's total debt is broken down as: (i) 0% insurance reserves; (ii) 63% deposits; (iii) 13% short term debt; (iv) 11%

long term debt; and (v) 13% other liabilities. Thus, we set the maturity of BAC's debt to $0\% \times 30 + 63\% \times 1 + 13\% \times 2 + 11\% \times 8 + 13\% \times 3 = 2.16$ years.

Admittedly, our assignment of maturities to various liability types is a bit ad-hoc. In an earlier version of the paper, we took an alternative two step estimation approach. First, we would fix a time to maturity and estimate the optimal parameters of the model for that time to maturity. We repeated this process for different time to maturities and chose the maturity that delivered the highest likelihood.[10] The downside of this approach is that the estimated maturity is difficult to interpret. In addition, changing ϕ in our model has a similar effect to changing τ; thus, we found it more economically sensible to assume reasonable maturities for each liability, and then treat the time to maturity as an exogenous input into the model. This also has the added attractiveness of incorporating time-variation in the effective maturity of debt, which is a pervasive feature in the data. In the Online Appendix, we show that our parameter estimates are not particularly sensitive to the choice of debt maturities (within reasonable ranges).

Asset Volatility as an Input to the Leverage Multiplier

From Equations (4) and (5), it is clear that we must also make a choice of what asset volatility $\sigma_{A,t}^{\tau}$ to input into the leverage multiplier. Recall that $\sigma_{A,t}^{\tau}$ is the expected cumulative volatility over the life of the option (e.g. the debt maturity). In the options pricing literature, the option value and option delta (and hence the leverage multiplier) depend on the total volatility over the life of the option — this is why we distinguish $\sigma_{A,t}^{\tau}$ from the "instantaneous" asset volatility $h_{A,t}$. With this in mind, we take two different approaches in computing $\sigma_{A,t}^{\tau}$. The first simply uses the unconditional variance implied by the GJR process for assets.[11] This means that the long-run cumulative asset volatility driving the leverage multiplier is constant, even though short run volatility $h_{A,t}$ is determined by a GJR process. The second approach is to use the GJR forecast for cumulative asset volatility over the life of the debt. This forecast naturally adjusts with the level of asset volatility $h_{A,t}$ at each date t and we provide an exact expression for it in Appendix C. In estimation, we use both approaches for $\sigma_{A,t}^{\tau}$ and choose the model that delivers the highest value of the likelihood function.

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[10]Note that in the model, the null that $\phi = 0$ implies that the time to maturity would be unidentified if treated as a free parameter in estimation. This is what motivated our search procedure.

[11]In terms of the parameters in Equation (5), this is $252 \times \tau_t \times \omega/(1 - \alpha - \gamma/2 - \beta)$.

Initial Conditions

In order to conduct maximum likelihood estimation, the recursive nature of the Structural GARCH model (Equation (5)) requires us to make some assumptions about initial conditions. To see this more clearly, suppose we are given a parameter set $\Theta = (\omega, \alpha, \gamma, \phi)$ and want to compute equity variance h_E at time $t = 2$. The Structural GARCH recursion implies that:

$$
\begin{aligned}
h_{E,2} &= LM_1 h_{A,2} \\
h_{A,2} &= \omega + \alpha \left(r_{A,1}\right)^2 + \gamma \left(r_{A,1}\right)^2 1_{r_{A,1}<0} + \beta h_{A,1} \\
&= \omega + \alpha \left(\frac{r_{E,1}}{LM_0}\right)^2 + \gamma \left(\frac{r_{E,1}}{LM_0}\right)^2 1_{r_{E,1}<0} + \beta h_{A,1} \\
LM_1 &= \left[LM^{BSM}\left(\frac{D_1}{E_1}, \sigma^\tau_{A,1}, \tau_1, r_1\right)\right]^\phi
\end{aligned}
$$

Here, $r_{E,1}, D_1/E_1, \tau_1, r_1$ are all directly observable at time $t = 1$. $\sigma^\tau_{A,t}$ depends on τ_t, the parameters Θ, and potentially $h_{A,1}$. Thus, the two variables that need initialized are $h_{A,1}$ and LM_0. We set $h_{A,1}$ equal to the unconditional asset variance implied by the GJR model, namely $\omega/(1 - \alpha - \gamma/2 - \beta)$. In addition, we set $LM_0 = 1$. To avoid the sensitivity of our estimated parameters to these choices, we throw out the first month's (21 days) worth of data when evaluating the likelihood function during the optimization. Obviously, we start the recursion from $t = 2$. It is also worth noting that we can recover asset returns (or what we interpret as asset returns) without directly observing them; this is accomplished simply by dividing current equity returns by the lagged value of the leverage multiplier.

3.2 Estimation Results

Table 1 contains some basic summary statistics on the sample of firms we consider. The average firm in our sample has a mean leverage ratio of around 8, though the standard deviation of 6.81 indicates there is fair amount of cross-sectional heterogeneity in firm leverage. Our sample also contains a pretty wide range of firms in terms of size, as measured by the average market value of equity. The 25th percentile of size in our sample is $2.5bn, whereas the 75th percentile is $19bn. Unsurprisingly, financial firms also have higher volatility than most firms. The average annualized equity volatility across

our sample is 40.3%, compared to a volatility of 20.1% for the CRSP Value-Weighted index over the same time period.

A summary of the parameter estimates from the Structural GARCH model are given in Table 2. Recall that in our model, the GJR parameters $(\omega, \alpha, \gamma, \beta)$ apply to each firm's asset return series. Using the median parameter values listed in Table 2, standard GARCH results imply that the median asset volatility is around 14% per year.[12] This is much less than the median equity volatility of 38% (Table 1), which reflects the fact that these firms have pretty high leverage.

Consistent with well-known features about equity volatility, the persistence of asset volatility is quite high, as evidenced by the fact that the estimated β's are all close to 0.9. Interestingly, 75 percent of our firms have a γ parameter of at least 0.046 and a t-statistic of at least 1.96. γ captures whether asset volatility responds differently to negative and positive news, and a positive γ implies that negative news raises future volatility more than positive news of equal magnitude. This is typically called volatility asymmetry and we will explore it in much more depth in Section 4.2.

The new parameter in our model is ϕ, and the median ϕ for the firms in our sample is 0.68. It is statistically different from zero (p-value ≤ 0.1) for about 60 percent of the firms in our sample. Keep in mind that our model collapses to a standard GARCH model for $\phi = 0$. These parameter estimates therefore suggest that effect of leverage on equity volatility is substantial for a large number of financial firms.

When $\phi = 1$, our leverage multiplier basically collapses to the Black-Scholes-Merton leverage multiplier. Importantly, $\phi = 1$ is contained within the 95 percent confidence band for 75 of the 91 firms in our sample (e.g. 82 percent of firms). This findings is consistent with Schaefer and Strebulaev (2008), who show that while the Merton (1974) model does poorly in predicting the level of credit spreads, it is successful in generating the correct hedge ratios across the capital structure of the firm. In our context, we interpret their conclusions and our estimated ϕ to mean that the Merton model does well in recovering the daily *return* of assets, even if it is not able to pinpoint the *level* of assets.

On the other hand, 22 of the 91 firms in our sample have a $\phi = 0$. For this subset of firms, leverage does not appear to help in explaining equity volatility. Unsurprisingly, this is because these firms have low leverage overall. Indeed, in a t-test of whether the

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[12]i.e. $\sqrt{252 \times \omega/(1 - \alpha - \gamma/2 - \beta)}$. Hence, the fact that our estimated ω's are small simply reflects the fact that asset volatility is less than equity volatility.

average leverage for firms with $\phi = 0$ is less than the leverage for firms with $\phi \neq 0$, we can reject the null that the two sets of firms have equal leverage (p-value $= 0.076$).[13] Perhaps more importantly, the standard errors on the point estimates for firms with $\phi = 0$ are quite large, as evidenced by the fact that 17 of these 22 firms also contain $\phi = 1$ in their 95 percent confidence interval. This is not surprising given that economic theory suggests ϕ is likely closer to one than it is to zero.[14] When focusing on firms with $\phi \neq 0$, the median point estimate for ϕ rises to 0.82 and 80 percent have a ϕ that is statistically different from zero (p-value ≤ 0.1). Unless otherwise noted, we exclude firms with $\phi = 0$ for the balance of the paper.

3.3   The Leverage Multiplier Across Firms and Time

One of the novelties of our model is a data-driven estimate of the leverage multiplier, which measures how leverage amplifies asset volatility into equity volatility. Figure 4 plots the lower quartile, median, and upper quartile of the cross-section of estimated leverage multipliers. This is done at each point in time. The median leverage multiplier hovers around 3, but rose to nearly 4 during the financial crisis. This means that, for the median firm, equity volatility is 3 to 4 times as large as asset volatility. Clearly there is a great deal of heterogeneity across firms in terms of leverage amplification. For instance, at the peak of the crisis, the lower quartile of firms had a leverage multiplier around 2, whereas the upper quartile of firms had a leverage multiplier of about 8.

To get a better sense of how leverage impacts volatility in the aggregate, we create an financial sector index of equity volatility, asset volatility, and the leverage multiplier. At each point in time, we take a weighted average across firms of our estimated (annualized) equity volatility series, where a firm's weight is determined by its pseudo-asset value. The pseudo-asset value of a firm is its market value of equity plus its book value of debt. We then apply this procedure to our estimated asset volatility series, as well as the leverage multipliers.

Panel A of Figure 5 plots these financial sector indices for our full sample period. One thing that stands out in the picture is that financial sector equity volatility peaked

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[13]Of the firms with $\phi = 0$, 18 percent of the firms have an average leverage less than 2. For the sample of firms with $\phi > 0$, only 6 percent of the firms have an average leverage of less than 2.

[14]For instance, in the case of riskless debt, the Modigliani-Miller theorem says that $\sigma_E = \frac{A}{E}\sigma_A$. In our context, this means the equity is deep in the money, so that the "delta" of the option converges to 1, the true leverage multiplier should be A/E, and thus $\phi = 1$.

at nearly 200% during the financial crisis. To put this in perspective, the popular VIX volatility index reached about 60% during the crisis. Another thing that jumps out is that the level of the financial sector leverage multiplier is consistently higher than the median leverage multiplier in our cross-section of firms (see Figure 4). This owes to the fact that the firms with large pseudo-asset values also have the large leverage multipliers. Interestingly, in the aftermath of the crisis the leverage multiplier has remained a bit higher than its pre-crisis level.

Panel B of Figure 5 provides a more detailed look at the asset volatility and the leverage multiplier of the financial sector during the financial crisis (2007-2009). It is clear that the rise in equity volatility for the aggregate financial sector began in the summer of 2007. However the rise in asset volatility did not really occur until late in 2008. Thus, the initial increase in equity volatility was driven by an increase in the leverage multiplier; this reflects both an increase in aggregate liabilities and a fall in equity valuations. After the fall of Lehman Brothers, asset volatility rose dramatically as well and the leverage multiplier continued to rise before stabilizing in the spring of 2009.

3.4   Additional Model Validation

We have already shown evidence that the Structural GARCH model outperforms a standard GJR model for the majority of firms in our sample. This follows directly from the fact that the Structural GARCH model nests a GARCH/GJR model. In order to provide additional model validation, we conduct a volatility forecast comparison using realized volatility as the forecasting target. In particular, we use the methods from Patton (2011) to compare our Structural GARCH model to a standard GJR model. Intuitively, our approach compares the two models based on the distance of their variance forecasts from the true conditional variance.

To develop these methods further, consider a single firm with a realized equity variance series rv_t. Patton (2011) argues that as long as rv_t is an unbiased estimate of the true conditional variance, then one can compare volatility forecasts by first computing a loss function at each point in time $L(rv_t, h_{mt})$. Here, h_{mt} denotes the forecasted variance from model m. For our purposes, m will either be the Structural GARCH model ($m = S$) or the GJR model ($m = G$). We will explicitly specify the loss function shortly, but loosely speaking, it is just a measure of distance between model's forecast and the

realized data.

Next, we compute the time-series of losses under each model and run the following regression:

$$L(rv_t, h_{Gt}) - L(rv_t, h_{St}) = c + \xi_t$$

There are two ways one can use this regression to carry out model comparison. The first is a simple model selection approach. In this case, the best model is the one with the smallest loss function. Thus, if $c > 0$, then the Structural GARCH has a smaller loss function, and vice versa.

The second way to compare the models is to statistically test whether they are equally close to the true realized variance rv_t. This is a straightforward application of the Diebold-Mariano-West test (e.g. Diebold and Mariano (2012)). In particular, we conduct a one-sided hypothesis test of the null that the two models have equal predictive accuracy $(c = 0)$, with the alternative being that the Structural GARCH outperforms the GJR model $(c > 0)$. We also test against the alternative that the GJR outperforms the Structural GARCH $(c < 0)$.

For our analysis, we consider two alternative loss functions:

$$
\begin{aligned}
L_{MSE}(rv_t, h_{mt}) &\equiv (rv_t - h_{mt})^2 \\
L_{QLIKE}(rv_t, h_{mt}) &\equiv \log(h_{mt}) + \frac{rv_t}{h_{mt}}
\end{aligned}
$$

We choose these loss functions because, as Patton (2011) shows, they are robust to noise in the realized variance proxy rv_t.

To measure realized variance, we use 5-minute intraday returns, which has been shown to be a much better proxy for the true conditional variance than daily squared returns (Andersen, Bollerslev, Diebold, and Labys (2003)). We obtain a "5M-RV" using the TAQ database and by applying the standard filters in the realized volatility literature. See Appendix D for more details. We are able to create reliable realized variance series for 42 out of the 91 firms in our sample; this means we conduct 42×2 comparisons in total, one for each firm and each loss function. In all of our Diebold-Mariano-West tests, we use a HAC covariance matrix to account for possible serial correlation and heteroscedasticity in the regression errors ξ_t. Table 3 contains a summary of the model selection analysis and the t-statistics from this set of tests.

In terms of model selection, the Structural GARCH model is preferable for a majority of the firms in our sample. For L_{QLIKE}, 57 percent of firms have a smaller loss function with Structural GARCH versus a GJR. This same number is 64 percent when using the L_{MSE} loss function.

For hypothesis testing under the L_{QLIKE} loss function, 14 of the 42 firms reject the null of equal predictive power in favor of the alternative hypothesis that the Structural GARCH outperforms the GJR volatility forecasts ($p \leq 0.05$). Still, 12 of the 42 firms show the reverse pattern in terms of favoring the alternative hypothesis that the GJR forecasts outperform those from the Structural GARCH ($p \leq 0.05$).

A clearer picture emerges when using the L_{MSE} loss function. For 16 of our 42 firms, hypothesis testing favors the predictive power of the Structural GARCH over the GJR. On the other hand, we can reject the null of equal predictive power in favor of the GJR model for only 3 of the 42 firms in our sample. Combined with the parameter estimates from Section 3.2 and the simple model selection criteria, these results suggest that the Structural GARCH model provides a meaningful improvement over the standard GARCH class of models in terms of measuring and forecasting volatility.

4   Applications

To demonstrate the usefulness of our model, we now explore two applications of the Structural GARCH: (i) systemic risk measurement and (ii) unpacking the causes of asymmetric volatility.

4.1   Systemic Risk Measurement

4.1.1   The Interaction Between Capital Structure and Future Risk

The recent financial crisis highlighted the need to quantify how future equity returns and risk are impacted by changes in capital structure. For instance, consider the U.S. government's Troubled Asset Relief Program (TARP) that was implemented during the financial crisis. Through TARP, the U.S. government purchased toxic assets and equity from a number of financial institutions. The purpose of this program was to reduce the risk of these institutions, but by how much?

Our model provides a framework to answer this question because it allows a firm's future equity return distribution to depend on its current capital structure. This is

a natural economic outcome, though one that is not explicitly present in most time-series models of volatility (e.g. the GARCH class). To illustrate, suppose we are standing on October 27, 2008 and considering a capital injection of $25 bn dollars into Bank of America (BAC). This is the amount that BAC would receive on the following day from the U.S. government. We can evaluate the impact that this injection would have on BAC's equity risk by conducting two simulations of the Structural GARCH model.[15] In the first simulation, we look at BAC's future equity return distribution using its current capital structure. In the second simulation, we study the same return distribution, but assume that BAC receives an equity injection of $25 bn. We use BAC's estimated Structural GARCH parameters for both simulations, so that the only difference across the simulations is the firm's initial capital structure. The simulation horizon ranges from one day to one month.[16]

Table 4 contains the simulated equity return quantiles for the "no injection" and the "injection" case. To save space, we report only the one-day ahead quantiles and the one-month ahead quantiles. It is clear from this simulation exercise how changing the capital structure of the firm alters its future equity return distribution. At both horizons, adding more capital lowers the volatility of equity returns and reduces the likelihood of extreme events. For instance, at a one-month horizon, a capital injection moves the one-percent quantile from -100% (bankruptcy) to -82.49%. Figure 6 makes this point sharper by focusing on the left tails of the two simulated distributions. With no injection, the probability of going bankrupt is 1.32%, but this probability shrinks by a factor of nearly three when adding capital to the firm. It is of course up to the regulator to assess the magnitude of this effect, and in turn, how large of a capital injection to provide — our model presents a way to quantify this tradeoff. We reiterate that in a standard GARCH setting this thought experiment would be useless because future returns are not influenced by today's capital structure.

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[15]This argument implicitly assumes that the market had not priced any potential government interventions into BAC's equity. While this seems unlikely, our goal is simply to highlight how changing the capital structure impacts the future equity return distribution.

[16]The estimated Structural GARCH parameters for BAC are: $\omega = 1.54 \times 10^{-7}$, $\alpha = 0.030$, $\gamma = 0.052$, $\beta = 0.936$, and $\phi = 0.746$. The equity value on 10/27/2008 was $93.6 bn, the face value of debt was $1562.2 bn, and the effective maturity of the debt was 2.38. We used a risk-free rate of 1.59%, as this was the 2 year constant maturity treasury yield on this date. When simulating the Structural GARCH model, we assume any paths where D/E exceeds 150 are bankruptcy paths; this alleviates the potential for explosive simulations. A threshold of 150 is very conservative, given that Lehman Brothers was effectively bankrupt when its leverage crossed 50.

4.1.2 Precautionary Capital

Based on the preceding discussion, we now introduce a new measure of systemic risk called precautionary capital. Precautionary capital asks the question: how much capital would a financial firm have to raise today in order to ensure with confidence c that it will not need to raise capital if there is a crisis? There are two differences between PCAP and other measures of capital shortfall such as SRISK (Brownlees and Engle (Forthcoming)). The first is that SRISK ask how much capital would be needed at the end of the stress period to allow the firm to continue to function. On the other hand, PCAP asks how much capital would it need today so that even under the stress of a crisis, it would have adequate capital. This question is the preventive question that both regulators and firm owners would like to answer. The second difference is that SRISK is asking for the median amount of capital that a firm would need to raise to continue to function whereas PCAP is asking how much capital would be needed to be highly confident that it will continue to function. In this case highly confident can be interpreted as 90% or 95% or even higher levels of confidence.

To develop this measure further, we use the following notation. Given initial levels of debt D_0 and equity E_0, we define the likelihood that future equity value E_T falls above any positive value x, conditional on a crisis:

$$f(x; E_0, D_0) := \mathbb{P}(E_T \geq x | crisis)$$

In general, we allow the function $f(\cdot)$ to depend on the initial value of leverage, though as we have emphasized many popular volatility models do not have this feature. Additionally, we build on Acharya, Pedersen, Philippon, and Richardson (Forthcoming) and Brownlees and Engle (Forthcoming) and define a crisis to be a 10 percent drop in the aggregate stock market over the next month. We also assume that debt is "sticky" in the sense that it cannot be altered over the next month.

Next, suppose that we want the value of equity to be above a fixed value E_T^* in a crisis. A natural candidate for this threshold derives from the fact that regulated financial institutions face capital requirements. For instance, if regulators want financial

institutions to have an equity-to-asset ratio of k, then:

$$k \quad = \quad \frac{E_T^*}{E_T^* + D_0}$$
$$\Leftrightarrow$$
$$E_T^* \quad = \quad \frac{kD_0}{1-k}$$

We can now solve for the amount of equity the firm would have to raise in order to have a level of confidence $c \in [0, 1]$ that it meets a capital requirement of k in a crisis. First, define E_0^* as the initial equity level that would deliver this confidence level. Formally, it is the E_0^* such that:

$$f(E_T^*; E_0^*, D_0) = c$$

Finally, we define precautionary capital as the difference between E_0^* and the true value of today's equity:

$$PCAP(k, c; \widehat{E}_0, D_0) := \max\left(E_0^* - \widehat{E}_0, 0\right) \tag{6}$$

where $\widehat{E}_0$ is today's actual equity level. We truncate $PCAP$ at zero because it is a measure of how much capital the firm needs today in order to survive a financial crisis. It is not designed to measure how much extra capital is available now.

Computing $PCAP$ with the Structural GARCH model involves solving a complicated nonlinear root problem. It requires us to first compute the quantile function of E_T as a function of E_0 and D_0, then to invert this function to solve for E_0^*. This is because the future return distribution depends on the current capital structure of the firm. Appendix E contains complete details for how we compute $PCAP$ under the Structural GARCH model.

The Capital Requirement k We set $k = 8\%$ (i.e. an asset-to-equity ratio of 12.5) because in calm times, well-managed financial firms had leverage ratios at about that level. For example in 2010, Wells Fargo — arguably the most conservatively managed large retail bank in the U.S. — averaged a capital ratio of $k = 10\%$. As another example, the asset-to-equity ratio at HSBC was about 13-to-1, for a capital ratio just under $k = 8\%$. Using a broader sample of 150 U.S. financial institutions, only 25 had capital ratios below 8% on average during 2010.

Financial Sector Precautionary Capital   Figure 7 plots the total $PCAP$ of all firms in our sample through time. The top panel shows financial sector $PCAP$ for the full sample, and the bottom panel focuses on its evolution through the financial crisis. For both plots, we target a confidence of 90% that a firm will meet a $k = 8\%$ capital requirement in a crisis.

Total $PCAP$ is relatively low for most of the early 2000s, but rises sharply in mid-2007. At the peak of the crisis, the total $PCAP$ of the financial sector reaches nearly \$2 trillion. Interestingly, after the crisis, $PCAP$ has not come close to returning to its pre-crisis levels. This likely derives from the fact that the financial sector leverage multiplier has also displayed the same pattern (see Figure 5).

The size of the U.S. TARP program was roughly \$700 billion, so the $PCAP$ numbers at the peak of the crisis seem large in comparison. There are at least two reasons why total $PCAP$ is so big. Total $PCAP$ gives the amount of equity that would be needed to be 90% sure that no firms fall below $k = 8\%$ in a crisis. This is not the same as the amount of capital needed to be sure that 90% of the firms meet their capital requirement. In addition, the systemic risk externality will mean that reducing the risk of one firm will reduce the risk of other firms, as well as in the broad economy. Hence, total $PCAP$ can be viewed as an upper bound of some sort on the ex-ante capital needs of the financial sector.

4.2   What Causes Volatility Asymmetry?

One nice feature of the Structural GARCH model is that it produces an estimate of asset volatility (h_{At} in the model). This enables us to say something about the cause of volatility asymmetry that is often observed in equity returns. More precisely, we say that a firm displays volatility asymmetry when negative equity returns predict higher future volatility, relative to positive equity returns of the same magnitude (Nelson (1991)). We measure volatility asymmetry in two ways. The first is with a simple correlation statistic and the second is with the estimated γ parameter from the GJR model (Section 4.2.2 provides more intuition).

There are two competing views for why we observe volatility asymmetry. The original explanation for this phenomena is what we call a mechanical leverage effect (Black (1976), Christie (1982)): declines in equity today mechanically increase leverage, and

because higher leverage is associated with more risk, future equity volatility also rises.[17] French, Schwert, and Stambaugh (1987) observe that risk premiums could also account for volatility asymmetry. In their alternative, a rise in future volatility raises the required return on equity, leading to an immediate decline in the stock price. We call this a risk premium effect. Our model let's us disentangle these two explanations.

4.2.1 Volatility Asymmetry: Simple Correlations

For a given time series x_t, the first way we quantify volatility asymmetry with a simple correlation, $\rho(|x_t|, x_{t-1})$. We compute this correlation using the equity returns of each firm in our sample and denote it by ρ_E. Table 5 contains some basic summary statistics for the cross-section of ρ_E. Consistent with the leverage effect, ρ_E is negative and statistically different from zero for a large majority of our firms. To get a sense of magnitude, ρ for the CRSP Value-Weighted equity index is about -0.11, so by this measure our set of firms does not seem to have as much asymmetry as the market.

A natural way to pinpoint how much leverage actually accounts for volatility asymmetry is to compare the size of the effect in asset returns versus equity returns. The Structural GARCH model provides a convenient way to accomplish this because it delivers us a daily asset return series for each firm. We quantify volatility asymmetry in asset returns using the same correlation statistic that we used with equity returns and we denote this by ρ_A. The top panel of Figure 8 is a scatter plot of each firm's ρ_E against its corresponding ρ_A. The plot basically tells the entire story of this exercise. Almost all of the points in the plot line up on the 45-degree line, indicating that the most of the volatility asymmetry seen in equity returns is still present in asset returns. It does not appear that leverage accounts for much of volatility asymmetry. Table 5 puts some more precise numbers to the plot. The median of the ratio of ρ_A/ρ_E across firms is 0.97, which we interpret to mean that leverage explains only 3% of volatility asymmetry. This conclusion is generally consistent with previous studies (e.g. Choi and Richardson (2015) or Hasanhodzic and Lo (2013)) that find leverage to play a small role in the negative correlation between future equity volatility and equity returns.

Using the asset returns from the Structural GARCH, we can also assess the impact that the risk premium channel has on volatility asymmetry. The risk premium feedback explanation can only hold for priced risk factors. In turn, firm-level asset returns

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[17]Asymmetry then arises because negative equity returns have a differential impact on leverage than positive equity returns of the same magnitude (Christie (1982)).

may display volatility asymmetry through simple exposure to these priced risk factors. One way to parse this out in the data is to look at the idiosyncratic asset returns of each firm. We define idiosyncratic returns as the residuals $\eta_{i,A,t}$ from the following daily time-series regression:

$$r_{i,A,t} = a_i + b_i \times r_{M,t} + \eta_{i,A,t}$$

where $r_{M,t}$ is the CRSP value-weighted equity index. As a first pass, we consider a one-factor model of returns (CAPM), but one could easily add additional pricing factors. We quantify the amount of volatility asymmetry in idiosyncratic asset returns as before, i.e. $\rho_I \equiv \rho(|\eta_{A,t}|, \eta_{A,t-1})$.

Table 5 indicates there is very little volatility asymmetry left in idiosyncratic asset returns. The median volatility asymmetry for equity returns was $\rho_E = -0.045$, but median asymmetry in idiosyncratic asset returns is basically zero at $\rho_I = -0.003$. The bottom panel of Figure 8 emphasizes the result visually, especially when comparing it to the top panel of the figure. In the top panel (volatility asymmetry for equity returns versus asset returns), most of the points lay tightly near the 45-degree line. In the bottom panel (volatility asymmetry for equity returns versus idiosyncratic asset returns), most of the points are loosely scattered above the 45-degree line and shrinking towards zero. From Table 5, we see that the median ratio of ρ_I/ρ_E is 0.17; combined with the fact that the median ρ_A/ρ_E is close to one, these results indicate that exposure to the aggregate stock market accounts for around $100 - 17 \approx 80$ percent of volatility asymmetry. This finding supports the risk premium explanation of French, Schwert, and Stambaugh (1987) for volatility asymmetry.

4.2.2 Volatility Asymmetry: γ from the GJR Model

The GJR model that we use for asset volatility $h_{A,t}$ provides a second way to measure asymmetric asset volatility. As a reminder, the variance recursion for asset volatility in the Structural GARCH model is given by:

$$h_{A,t+1} = \omega_A + \alpha_A r_{A,t}^2 + \gamma_A r_{A,t}^2 1_{r_{A,t}<0} + \beta_A h_{A,t}$$

The response of tomorrow's asset variance $h_{A,t+1}$ to today's asset news is summarized nicely by the following derivative:

$$\frac{\partial h_{A,t+1}}{\partial r_{A,t}^2} = \begin{cases} \alpha_A & \text{if } r_{A,t} \geq 0 \\ \alpha_A + \gamma_A & \text{if } r_{A,t} < 0 \end{cases}$$

The parameter γ_A captures the potentially heterogeneous response of volatility to positive and negative news. When $\gamma_A > 0$, negative news raises variance more than positive news of the same magnitude. We therefore use γ_A to quantify volatility asymmetry in asset returns. Most of the subsequent analysis mirrors our analysis in Section 4.2.1, except we study how γ changes when looking at equity returns, asset returns, and idiosyncratic asset returns.

To start, we measure volatility asymmetry in equities by estimating a GJR model for each firm's equity returns. With some abuse of notation, we call this γ_E. As Table 5 shows, there is a quite a bit of volatility asymmetry in the equity returns for the firms in our sample. Nearly 90 percent of our firms have a γ_E that is different from zero at a 10 percent confidence level, with the median $\gamma_E = 0.081$. As a point of comparison, γ_E for the CRSP Value-Weighted equity index is 0.16.

γ_A is estimated from the Structural GARCH model, and we use the subscript A to make clear that this parameter measures volatility asymmetry for asset returns. The top panel of Figure 9 plots γ_A against γ_E. Most of the points on the plot lie close to, yet underneath of, the 45-degree line. This is a simple way of seeing that leverage does account for some of the volatility asymmetry in equity returns, but not very much of it. Table 5 confirms the intuition of the plot. The median γ_A is 0.07, so less than the median γ_E.[18] If leverage did account for asymmetry, the point estimate for γ_A would be much lower than γ_E. The median ratio of γ_A/γ_E suggests that only about 14 percent of volatility asymmetry in equity returns comes from leverage.

Next, we assess how much volatility asymmetry comes from exposure to priced risk factors. For each firm, we fit a GJR model to idiosyncratic asset returns, and collect the asymmetry parameter (denoted by γ_I). As before, we define idiosyncratic asset returns as the residuals from a regression of asset returns on the CRSP value-weighted equity index. The logic is the same as before: if exposure to priced risk factors accounts

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[18]The reason that the median γ_A does not line up with those reported in Table 2 is that we have excluded firms with $\phi = 0$.

for volatility asymmetry, then the idiosyncratic asset returns should no longer display asymmetry.

The bottom panel of Figure 9 plots γ_I against γ_E. Compared to the top panel, the points in the plot lie much further below the 45-degree line, indicating that idiosyncratic asset returns display far less asymmetry than equity returns. From Table 5 we see that the median γ_I is 0.015, or about 20 percent of the median γ_E. In fact, only 13 percent of firms in our sample have a γ_I that is significantly different than zero at a 10 percent confidence level. Combined with our analysis in Section 4.2.1, these results imply that most of the volatility asymmetry in equity returns comes through exposure to priced risk factors, as opposed to leverage itself; this is true regardless of how one measures volatility asymmetry. This conclusion also echoes Bekaert and Wu (2000), who find that the risk premium channel is the primary determinant of volatility asymmetry in their sample of Japanese stocks.

5 Conclusion

This paper has provided a time-series model of equity volatility that accounts for capital structure. Our Structural GARCH model blends the contingent claims view of Merton (1974) with GARCH-style models of Engle (1982) and Bollerslev (1986). In doing so, we are able to quantify the impact that capital structure has on future risk, which is a particularly important question in the wake of the 2008 financial crisis.

We applied the model in two different settings, asymmetric volatility and systemic risk measurement, though there are a variety of other settings in which our model can be used. For example, our analysis has centered on using structural models of credit to understand equity volatility dynamics; however, structural models of credit also make strong predictions for the volatility of credit spreads. Our framework may provide a better way to measure credit spread or credit default swap volatility by using the volatility-leverage connection highlighted in this paper, as well as observable equity shocks. In terms of systemic risk, it would be useful to build an asset pricing model that trades off raising precautionary capital today versus raising equity capital in a crisis. The Structural GARCH model is an econometric tool that can be used to explore all of these ideas.

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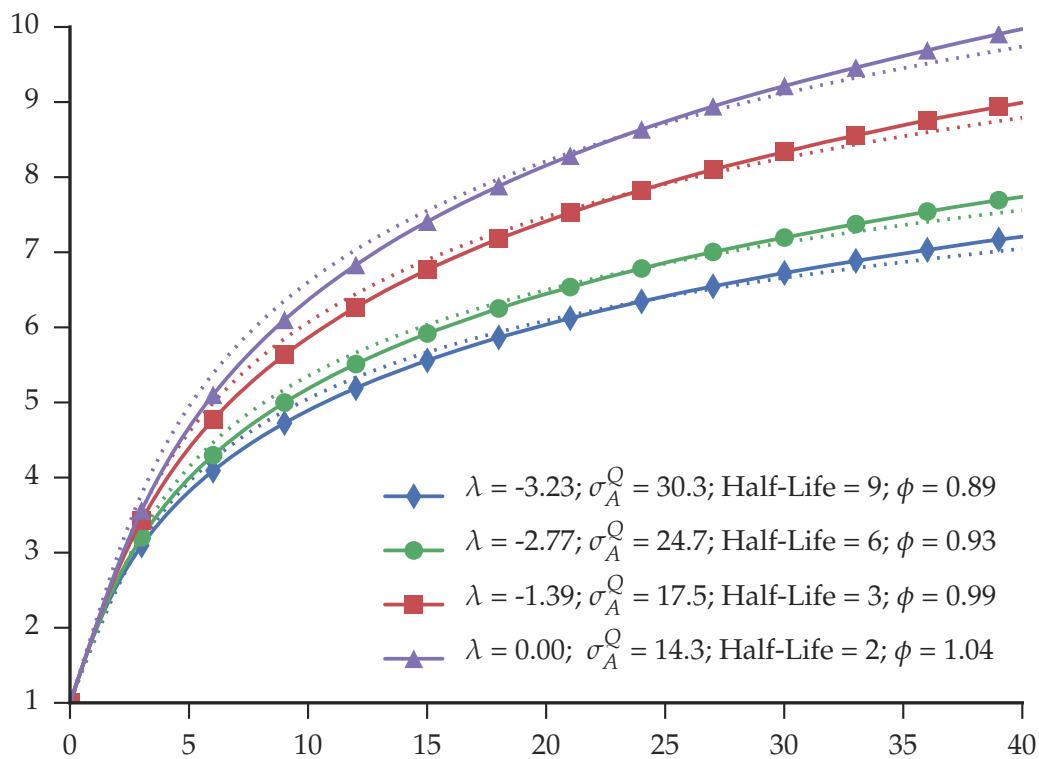Figure 1: The Leverage Multiplier in Various Option Pricing Models
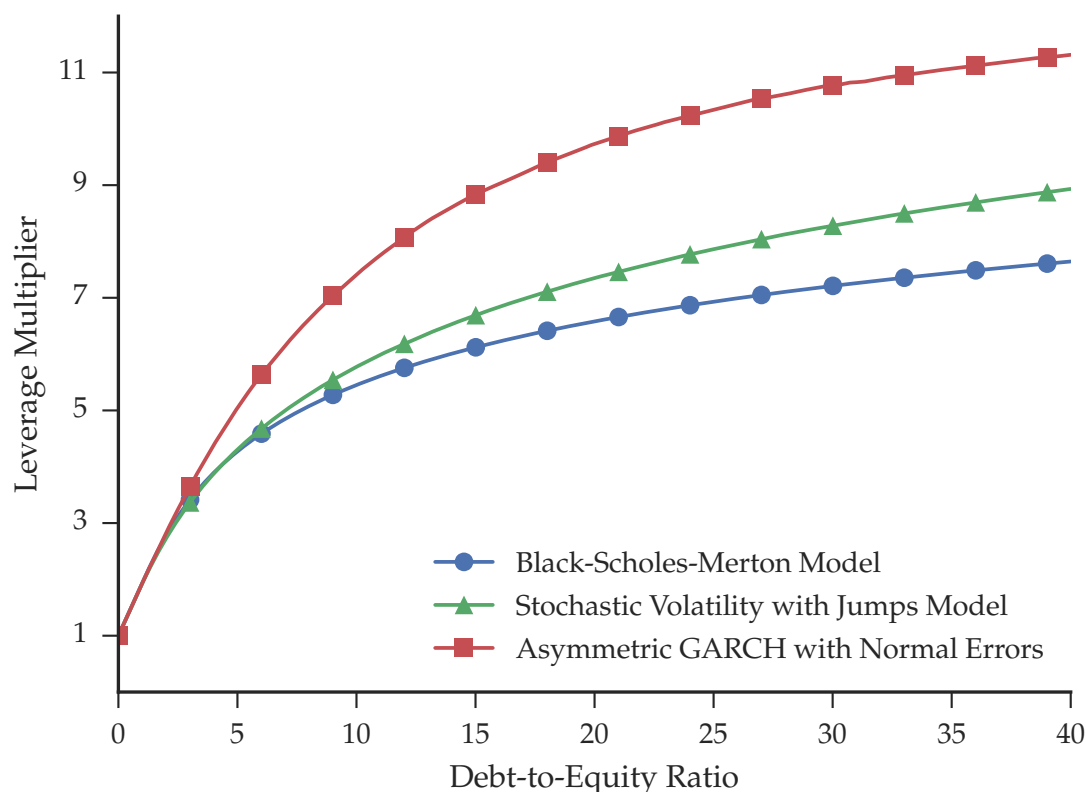
Notes: This figure plots the leverage multiplier in (i) the Black-Scholes-Merton model; (ii) a more general option pricing model where the underlying asset process has stochastic volatility and experiences jumps (e.g. Bakshi, Cao, and Chen (1997)); and (iii) a model where asset variance follows an asymmetric GARCH model (Glosten, Jagannathan, and Runkle (1993)) with normally distributed innovations. For all models, the interest rate $r = 0.03$, the time to maturity $\tau = 2$ years, and the total annualized volatility of assets is set to 17.5%. For the Bakshi, Cao, and Chen (1997) model: the speed of volatility mean reversion $\kappa = 2.77$; the correlation between volatility shocks and asset return shocks $\rho = -0.7$; long-run average asset variance $\theta = 0.013$; annualized volatility of asset variance $\sigma_v = 17.5\%$; jump intensity $\lambda_J = 0.4$; average jump size $\mu_J = -0.1$; and average jump volatility $\sigma_J = 0.15$. The current level of volatility is always set at its long-run average. In the GARCH model, asset variance h_A evolves as: $h_{A,t+1} = \omega + \alpha r_{A,t}^2 + \gamma r_{A,t}^2 1_{r_{A,t}<0} + \beta h_{A,t}$, with $\omega = 4.86 \times 10^{-7}, \alpha = 0.022, \gamma = 0.18, \beta = 0.884$. All parameters are stated in risk-neutral space.

Figure 2: How Our Proposed Leverage Multiplier Fits Other Option Pricing Settings
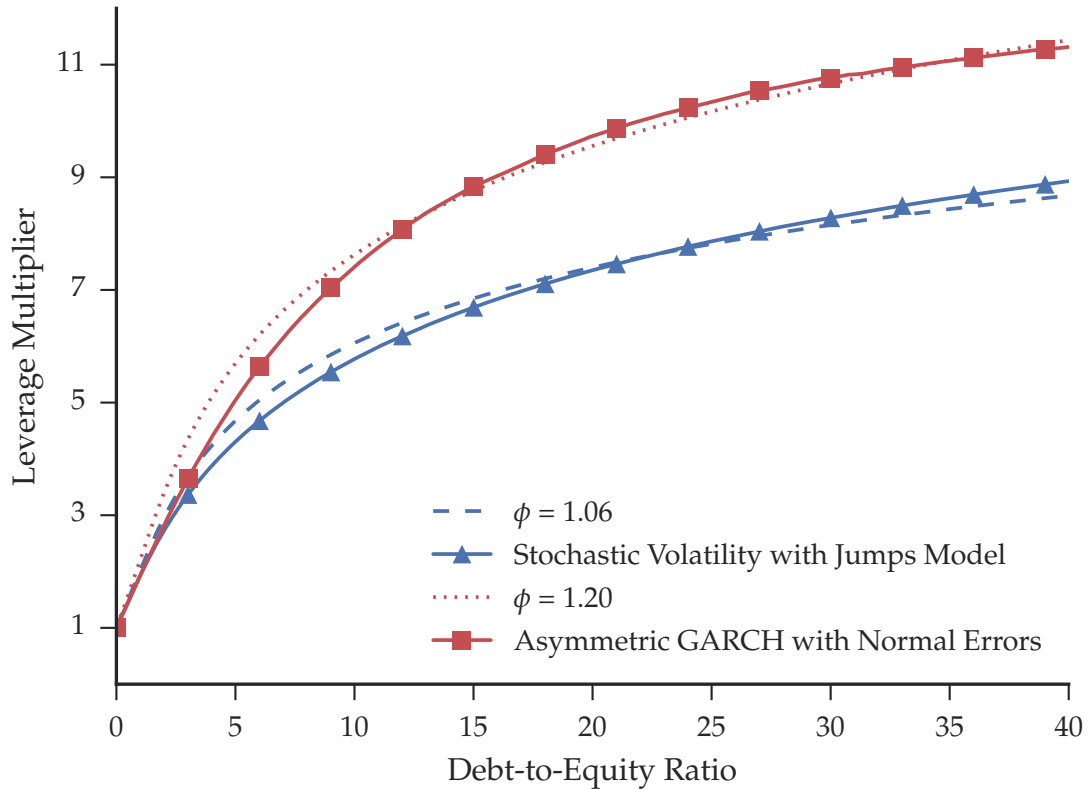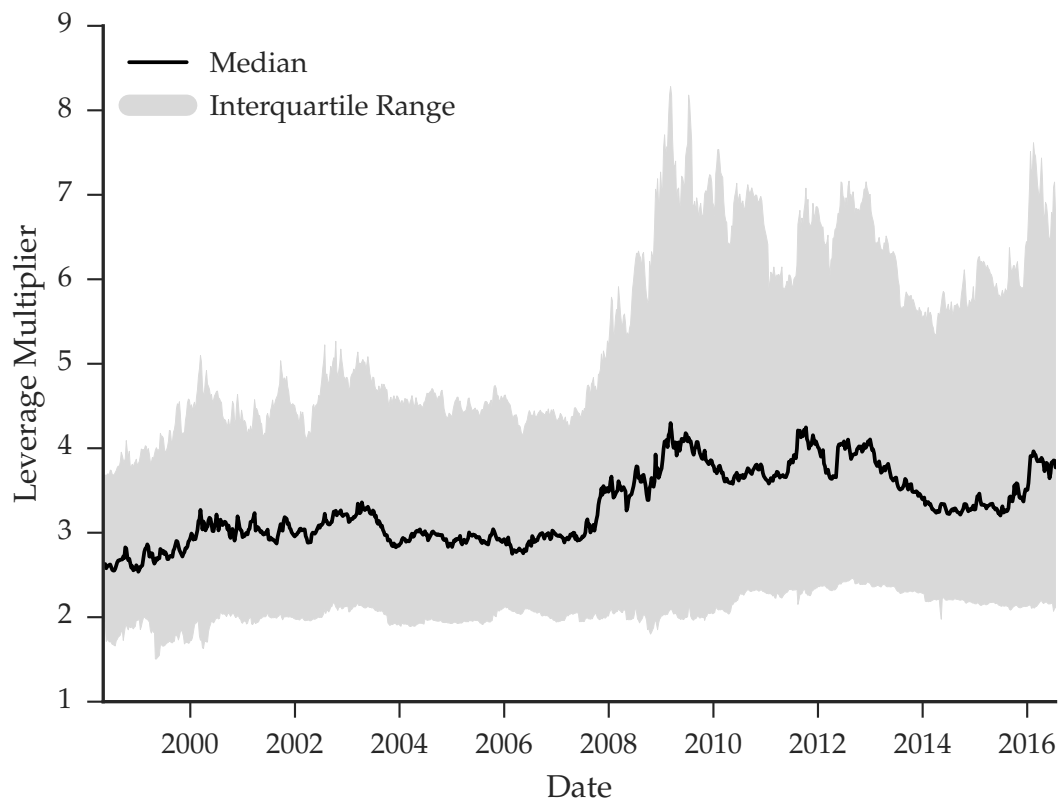

Notes: This figure plots the leverage multiplier in (i) a continuous-time model where the underlying asset process has stochastic volatility and experiences jumps (e.g. Bakshi, Cao, and Chen (1997)) and (ii) a discrete-time model where assets follow a GJR model with normally distributed errors. For both models, the interest rate $r = 0.03$, the time to maturity $\tau = 2$ years, and the total annualized volatility of assets is set to 17.5%. For the Bakshi, Cao, and Chen (1997) model: the speed of volatility mean reversion $\kappa = 2.77$; the correlation between volatility shocks and asset return shocks $\rho = -0.7$; long-run average asset variance $\theta = 0.013$; volatility of asset variance $\sigma_v = 17.5\%$; jump intensity $\lambda_J = 0.4$; average jump size $\mu_J = -0.1$; and average jump volatility $\sigma_J = 0.15$. The current level of volatility is always set at its long-run average. For the GJR model, risk neutral asset variance evolves according to: $h_{A,t+1} = \omega + \alpha r_{A,t}^2 + \gamma r_{A,t}^2 1_{r_{A,t}<0} + \beta h_{A,t}$. We set $\omega = 4.86 \times 10^{-7}, \alpha = 0.022, \gamma = 0.18, \beta = 0.884$. For both models, we toggle ϕ in our proposed leverage multiplier function in order to match each model's leverage multiplier. This is simply to demonstrate that our parameterization of the leverage multiplier can accommodate a variety of different option pricing settings.

Figure 3: Volatility Risk Pricing and Our Proposed Leverage Multiplier

Notes: This figure plots the leverage multiplier in the Heston (1993) model for a variety of parameter values governing the pricing of volatility risk. Under the physical measure, we set $\kappa = 4.16$, $\theta = 0.0204$, $\sigma_v = 0.175$, $\rho = -0.7$. These parameters correspond to a half-life for asset volatility shocks of 2 months, long-run asset volatility of 14.3%, and a volatility of asset volatility of 17.5%. We then compute the leverage multiplier for four different values of the price of volatility risk λ. In all cases, the interest rate $r = 0.03$ and the time to maturity $\tau = 2$ years. The current level of physical volatility is always set at its long-run average. For each parameter combination, we toggle ϕ in our proposed leverage multiplier function in order to match the leverage multiplier implied by that parameter combination. The solid lines with markers are the true leverage multiplier under the Heston (1993) model, and the dotted lines show our approximation.
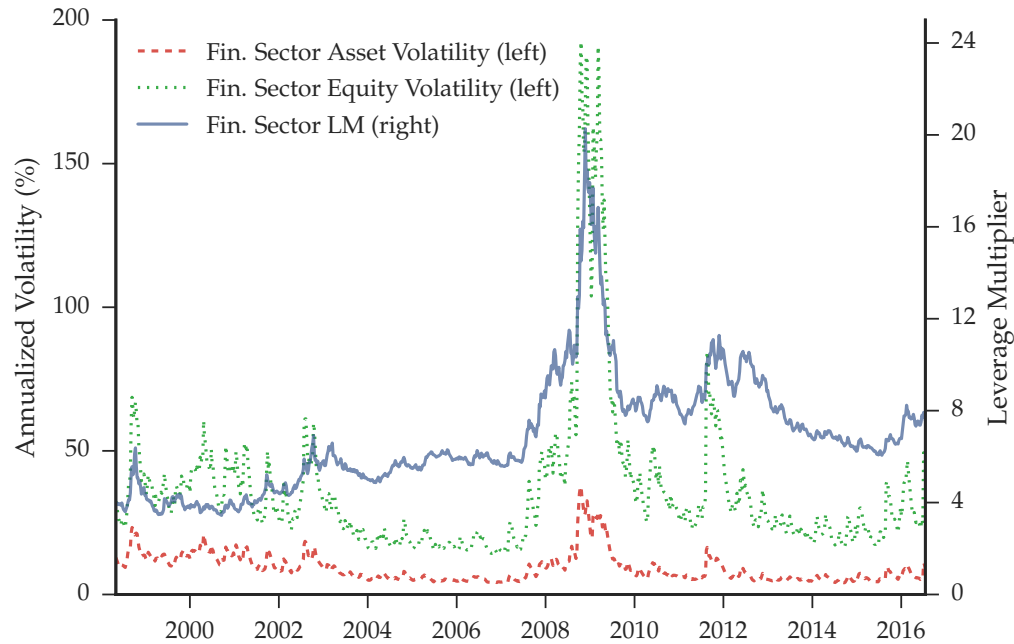
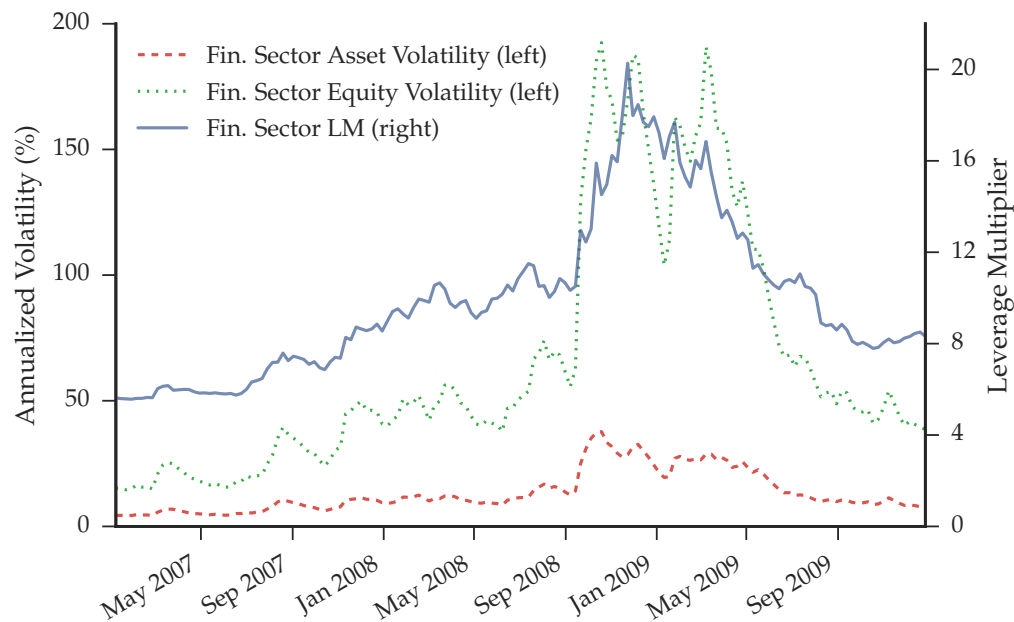Figure 4: How Does the Leverage Multiplier Vary Across Firms and Time?

Notes: This figure plots the leverage multiplier across firms and through time. We focus on firms for which $\phi \neq 0$ because these are the firms that have a non-constant leverage multiplier. For each firm, we take the estimated daily leverage multiplier and resample it at a weekly frequency by averaging within each week. The solid line is the cross-sectional median within each week, and the shaded bars represent the interquartile range.

Figure 5: Volatility and the Leverage Multiplier of the Financial Sector

Panel A (Full Sample):

Panel B (Financial Crisis):

Notes: This figure plots the equity volatility, asset volatility, and the leverage multiplier of the financial sector. We focus on firms for which $\phi \neq 0$ because these are the firms that have a non-constant leverage multiplier. For each firm, we take the estimated daily volatility or leverage multiplier series and resample it at a weekly frequency by averaging within each week. We then create an aggregate financial sector index at each point in time by taking an asset weighted-average across firms, where assets are proxied for by the market value of equity plus the face value of outstanding debt. We smooth the book values of debt using an exponential average (see Section 3.1). The top panel is for the full sample, the bottom panel is from January 2007 through December 2009.

Figure 6: The Impact of Capital Injections on One-Month Crash Likelihoods
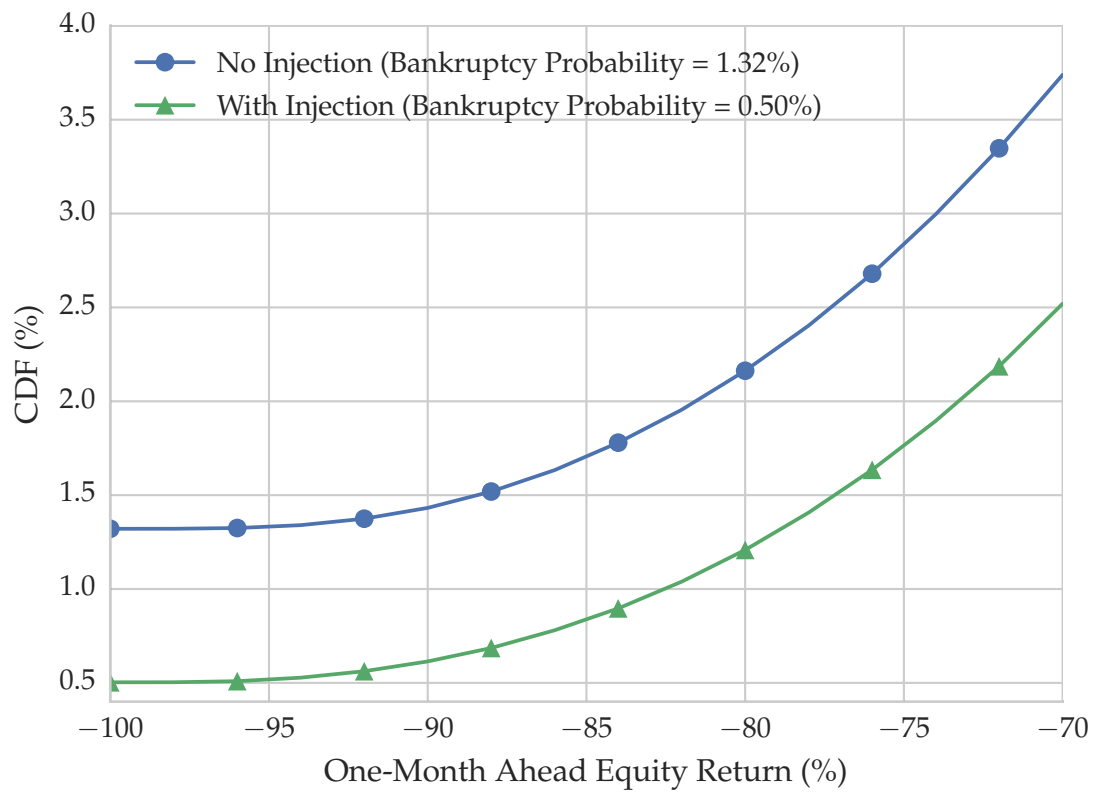
Notes: This figure plots the simulated cumulative equity return distribution for Bank of America (BAC), with a focus on the left tail of the distribution. For the simulations, we consider two different models. The first model is the standard Structural GARCH model and the second model is the Structural GARCH model where BAC has received an equity injection. All of the models are simulated using data as of 10/27/2008. The smoothed debt amount on this date was $1562.2 bn and the market value of equity was $93.6 bn. The amount of the injection corresponds to what BAC received from the U.S. government on 10/28/2008. For both models, we use the same set of standard normal shocks and simulate 10 million paths out to a horizon of one-month.

Figure 7: Precautionary Capital for the Financial Sector ($k = 0.08$)

Panel A (Full Sample):
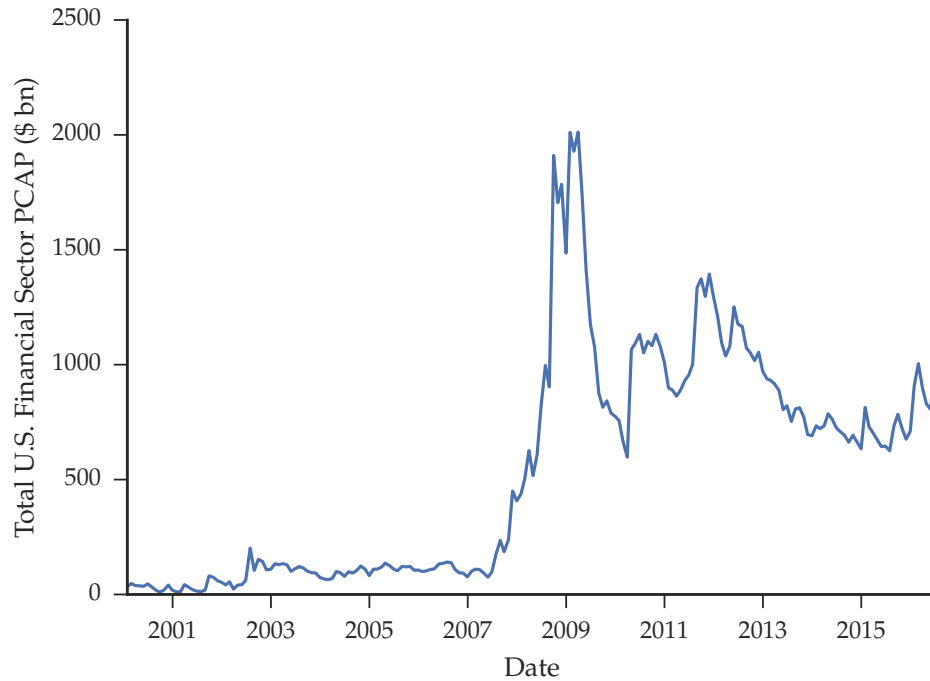
Panel B (Financial Crisis):
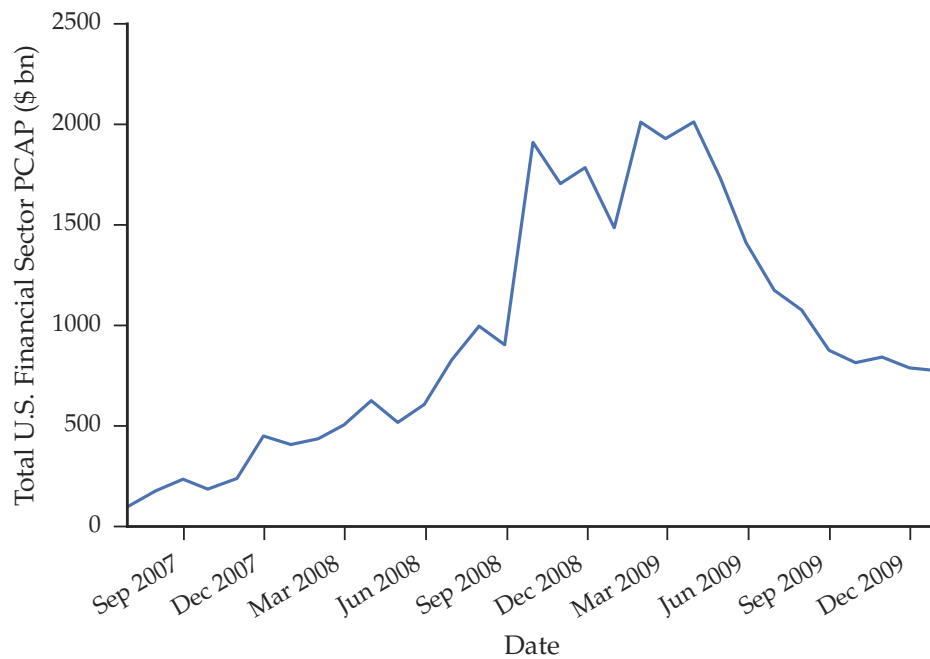
Notes: Panel A of this figure plots total precautionary capital for all financial firms in our sample. Precautionary capital is defined precisely in Section 4.1.2, and measures the amount of equity that a firm would need to raise today in order to ensure that it meets a capital ratio of k during a crisis, with an arbitrary confidence level. We define a crisis has a drop in the aggregate stock market of 10% over the next month. We target a capital ratio of $k = 8\%$ in a crisis and a 90% confidence level. The top panel shows precautionary capital for the full sample, and the bottom panel shows it during the financial crisis. Full details on how we compute precautionary capital are contained in Appendix E.

Figure 8: Volatility Asymmetry Measured with Simple Correlations

Panel A (Equity Versus Assets Returns):
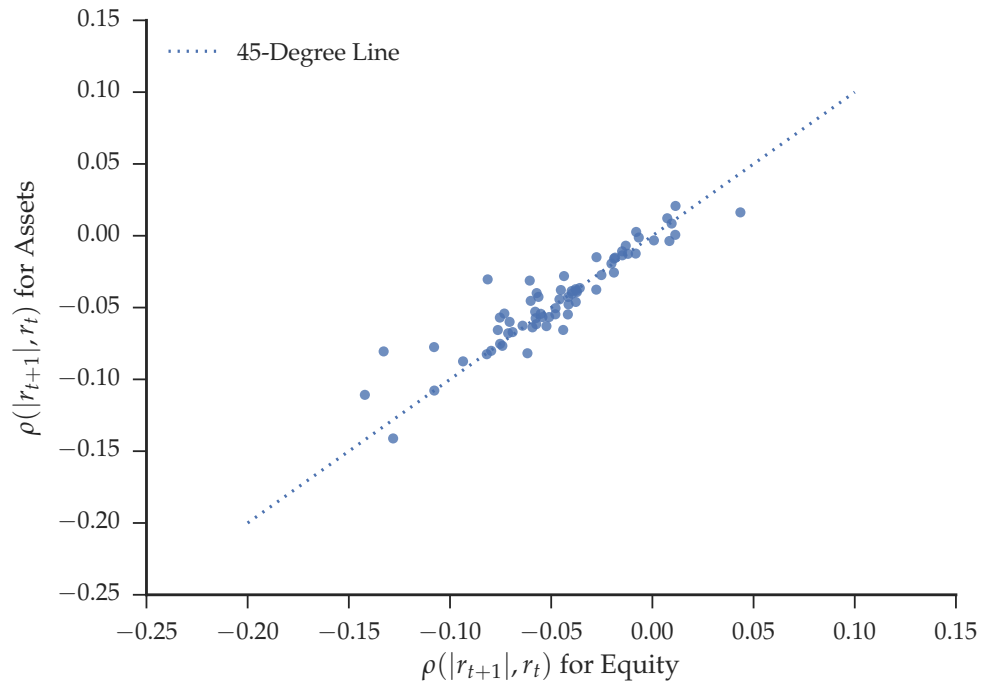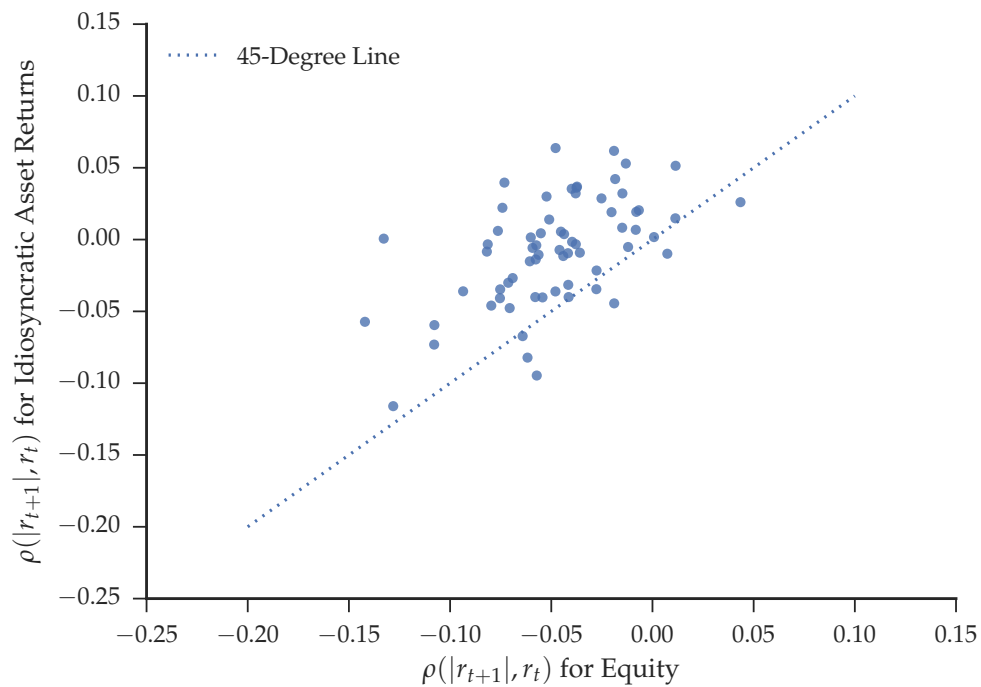
Panel B (Equity Versus Idiosyncratic Asset Returns):

Notes: This figure plots a measure of volatility asymmetry in equity returns against the same measure in: (i) asset returns (Panel A) and (ii) idiosyncratic asset returns (Panel B). Volatility asymmetry is measured as the correlation between the absolute value of returns at time t and returns at time $t-1$. Asset returns are computed using the leverage multiplier from the Structural GARCH model, and are simply equity returns divided by the lagged leverage multiplier. Idiosyncratic asset returns are defined as the residuals from a regression of asset returns on the CRSP value-weighted index.

Figure 9: Volatility Asymmetry Measured from the GJR Model
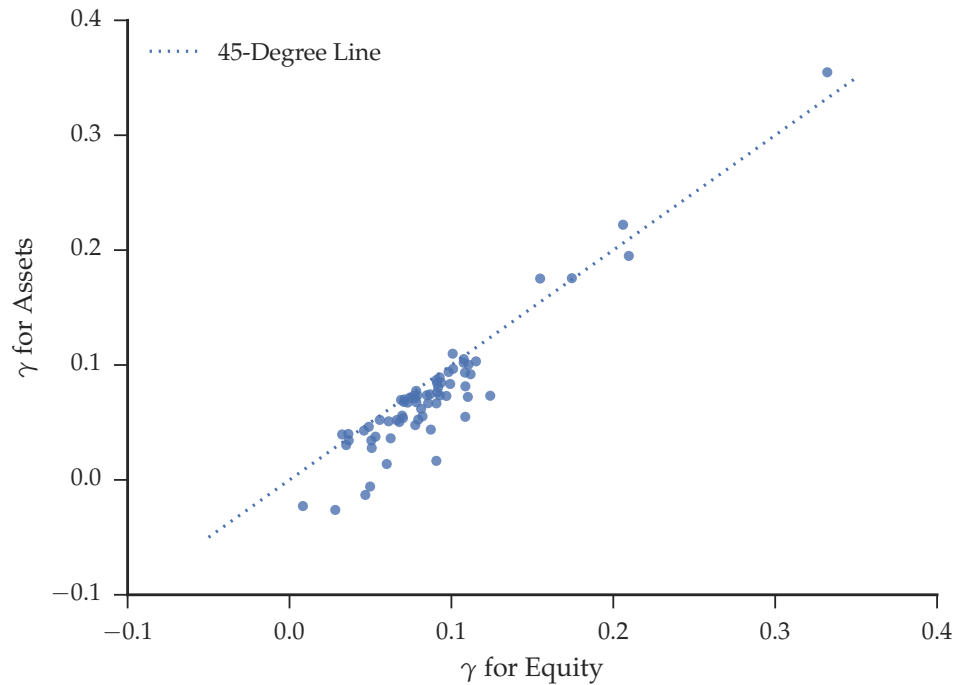
Panel A (Equity Versus Assets Returns):
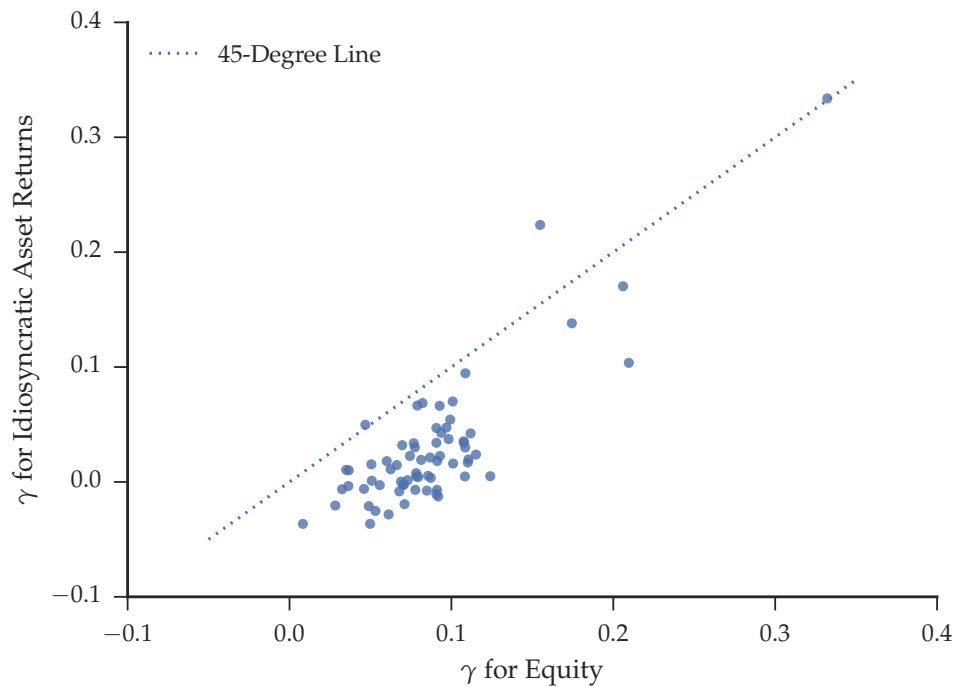
Panel B (Equity Versus Idiosyncratic Asset Returns):

Notes: This figure plots a measure of volatility asymmetry in equity returns against the same measure in: (i) asset returns (Panel A) and (ii) idiosyncratic asset returns (Panel B). Volatility asymmetry is defined as the parameter in the Glosten, Jagannathan, and Runkle (1993) model that measures the response of variance to negative return news (γ). Asset returns are computed using the leverage multiplier from the Structural GARCH model, and are simply equity returns divided by the lagged leverage multiplier. Idiosyncratic asset returns are defined as the residuals from a regression of asset returns on the CRSP value-weighted index.

Table 1: Cross-Sectional Summary Statistics

	D/E	Market Cap ($mm)	Ann. Volatility (%)	# Daily Obs.
Mean	8.04	19,535	40.3	4,040
Std. Deviation	6.81	33,958	11.0	988
Min	0.03	631	23.3	630
25-percentile	4.46	2,456	31.9	4,005
50-percentile	5.67	5,411	38.3	4,584
75-percentile	9.23	19,210	47.2	4,584
Max	30.77	169,843	82.0	4,584
N	91	91	91	91

Notes: This table presents cross-sectional summary statistics across the firms in our sample. For each firm and each variable, we take the time-series mean. We then compute cross-sectional summary statistics across firms. D/E is the smoothed book value of debt (see Section 3.1) divided by the market value of equity.

Table 2: Estimated Parameters and t-statistics of the Structural GARCH Model

		ω	α	γ	β	ϕ
Mean:	Point Estimate	2.74e-6	0.047	0.074	0.902	0.738
	t-statistic	1.23	2.76	2.83	72.07	3.42
25-percentile:	Point Estimate	2.66e-7	0.023	0.046	0.884	0.054
	t-statistic	0.80	1.98	1.96	48.01	0.07
50-percentile:	Point Estimate	7.30e-7	0.041	0.071	0.914	0.680
	t-statistic	1.12	2.91	3.01	71.24	2.36
75-percentile:	Point Estimate	4.47e-6	0.063	0.094	0.931	1.003
	t-statistic	1.57	3.57	3.84	91.76	4.64
% of Firms with $|t| \geq 1.64$		22.0	81.3	82.4	100	60.4

Notes: This table presents summary statistics across firms of the point estimates from the Structural GARCH model. The full specification of the Structural GARCH model can be found in Equation (5). The total number of firms in our sample is 91, and a full list can be found in INSERT. For each firm, we estimate the model using daily data. Full details of the estimation procedure can be found in Section 3. All reported t-statistics are QMLE robust (Bollerslev-Wooldridge).

Table 3: Volatility Forecast Comparison

	Loss Function	
	MSE	QLIKE
Mean of t-statistic	0.83	0.78
25-percentile of t-statistic	-0.67	-1.79
50-percentile of t-statistic	1.42	0.31
75-percentile of t-statistic	2.02	2.85
# of firms with t-statistic ≥ 1.64	16	14
# of firms with t-statistic ≤ -1.64	3	12
Model Selection: # of firms that favor Structural GARCH	27	24
N	42	42

Notes: This table presents results from Diebold-Mariano-West tests of equal predictive accuracy of realized volatility for the Structural GARCH model and a simple GJR model (Glosten, Jagannathan, and Runkle (1993)). The Diebold-Mariano-West is a daily time-series regression of the difference between the loss functions using the Structural GARCH and the simple GJR model onto a constant. The two loss functions we consider for the tests are the mean-squared-error (MSE) and quasi-likelihood function (QLIKE) from Patton (2011). We report the t-statistic on the point estimate of the constant. A t-statistic greater than 1.64 indicates a rejection of the null of equal predictive accuracy in favor of the alternative hypothesis that the Structural GARCH outperforms the GJR. A t-statistic less than -1.64 rejects the null of equal predictive accuracy in favor of the GJR model. For all firms, we measure daily realized variance using a 5-minute intraday measure of realized variance. Conditional on data availability, we run the Diebold-Mariano-West test for all firms in our sample that do not have $\phi = 0$ in the Structural GARCH. Additionally we remove realized variances where the number of trades on a day was in the bottom 5% of the sample of all firms. For each test, we use HAC standard errors . Model selection indicates the number of firms where the point estimate from the Diebold-Mariano-West test was positive (i.e. Structural GARCH outperforms the GJR).

Table 4: Bank of America: Simulated Equity Return Quantiles

Quantile (%)	1-day Ahead Return (%)		1-month Ahead Return (%)	
	No Injection	Injection	No Injection	Injection
1	-24.57	-22.13	-100.00	-82.49
5	-18.08	-16.21	-64.64	-59.38
10	-14.39	-12.87	-50.14	-46.26
25	-7.86	-7.00	-23.75	-22.33
50	0.00	0.00	9.45	7.34
75	8.51	7.51	47.56	40.88
90	16.79	14.76	87.44	75.52
95	22.05	19.33	114.44	98.74
99	32.57	28.41	173.19	148.75

Notes: This table presents simulated quantiles for Bank of America (BAC) equity returns under two different models and two different horizons (one day and one month ahead). The first model is the Structural GARCH (SGJR) model and the second model is the Structural GARCH model where BAC has received an equity injection (SGJR-Inject). All of the models are simulated using data as of 10/27/2008. The smoothed debt amount on this date was $1562.2 bn and the market value of equity was $93.6 bn. The amount of the injection corresponds to what BAC received from the U.S. government on 10/28/2008. For both models, we use the same set of standard normal shocks and simulate 10 million paths.

Table 5: Volatility Asymmetry

	Lower Quartile	Median	Upper Quartile	% with $p \leq 0.1$	N
Simple Correlations:					
ρ_E	-0.064	-0.045	-0.019	71.0	69
ρ_A	-0.062	-0.043	-0.016	69.6	69
ρ_I	-0.038	-0.003	0.022	52.2	69
ρ_A/ρ_E	0.76	0.97	1.06		68
ρ_I/ρ_E	-0.55	0.17	0.66		66
GJR Model:					
γ_E	0.062	0.081	0.099	89.9	69
γ_A	0.046	0.070	0.084	82.6	69
γ_I	-0.003	0.015	0.035	13.0	69
γ_A/γ_E	0.73	0.86	0.95		68
γ_I/γ_E	-0.04	0.20	0.40		68

Notes: This table presents a cross-sectional summary of the strength of volatility asymmetry across the firms in our sample. We consider only those firms who have a $\phi \neq 0$ in the Structural GARCH model (Equation (5)). We quantify asymmetry in two ways. First, for a given return series, x_t, we quantify the leverage effect as the correlation $\rho(|x_t|, x_{t-1})$. Second, we use the parameter in the Glosten, Jagannathan, and Runkle (1993) model (GJR) that measures the response of variance to negative return news (γ). For each firm, we measure volatility asymmetry for that firm's equity returns (subscript E), asset returns (subscript A), and idiosyncratic asset returns (subscript I). For asset returns, our second measure of volatility asymmetry is the estimated γ parameter in the Structural GARCH model. Asset returns are computed using the leverage multiplier from the Structural GARCH model, and are simply equity returns divided by the lagged leverage multiplier. Idiosyncratic asset returns are defined as the residuals from a regression of asset returns on the CRSP value-weighted index. When computing the ratio of parameters, we discard firms whose ratio is greater than 5 in absolute value in order to avoid the impact of outliers. The column "% with $p \leq 0.1$" indicates the percent of firms where the parameter is statistically different than zero with 10% confidence.

A   Appendix: Full Derivation of the Leverage Multiplier

A.1   A General Model of Asset Returns

In this appendix, we derive the relationship between equity volatility and asset volatility that is used in the main text. To start, define the equity value as follows:

$$E_t = f\left(A_t, D_t, \sigma_{A,t}, J_A, N_A, \tau_t, r_t; \Theta_p, \Theta_r\right) \tag{7}$$

where $f(\cdot)$ is an unspecified call option function, A_t is the current market value of assets, D_t is the current book value of outstanding debt, $\sigma_{A,t}$ is the (potentially stochastic) volatility of the assets. τ_t is the life of the debt, and finally, r_t is the annualized risk-free rate at time t. Additionally, J_A and N_A are processes that describe discontinuous jumps in the underlying assets. Θ_p is a vector of parameters that govern the evolution of assets, asset volatility, and asset jumps under the physical $\mathbb{P}$-measure. Θ_r is a vector of parameters that describe the pricing of risks and derives from the underlying preference parameters that enter the stochastic discount factor in the economy. Next, we specify the following generic process for assets and variance:

$$
\begin{aligned}
\frac{dA_t}{A_t} &= [\mu_A(t) - \lambda\mu_J]dt + \sigma_{A,t}dB_A(t) + J_A dN_A(t) \\
d\sigma_{A,t}^2 &= \mu_v(t, \sigma_{A,t})dt + \sigma_v(t, \sigma_{A,t})dB_v(t)
\end{aligned}
\tag{8}
$$

where $dB_A(t)$ is a standard Brownian motion. $\sigma_{A,t}$ captures time-varying asset volatility. We also capture potential jumps in asset values via J_a and $N_A(t)$. $\log(1 + J_A) \sim N\left(\log[1 + \mu_J] - \sigma_J^2/2, \sigma_J^2\right)$ and N_t is a Poisson counting process with intensity λ. The relative price jump size J_A determines the percentage change in the asset price caused by jumps, and the average asset jump size is μ_J. We assume the jump size, J_A, is independent of $N_A(t)$, $B_A(t)$, and $B_v(t)$. Similarly, the asset Poisson counting process $N_A(t)$ is assumed to be independent of $B_A(t)$ and $B_v(t)$. We allow an arbitrary instantaneous correlation of ρ_t between the shock to asset returns, $dB_A(t)$, and the shock to asset volatility, $dB_v(t)$.

The instantaneous return on equity is computed via simple application of Itō's Lemma for Poisson processes:

$$
\begin{aligned}
\frac{dE_t}{E_t} &= \Delta_t \frac{A_t}{D_t}\frac{D_t}{E_t} \cdot \frac{dA_t}{A_t} + \frac{\nu_t}{E_t} \cdot d\sigma_{A,t} + \frac{1}{2E_t}\left[\frac{\partial^2 f}{\partial A_t^2}d\langle A\rangle_t + \frac{\partial^2 f}{d(\sigma_{A,t})^2}d\left\langle \sigma_A^f\right\rangle_t + \frac{\partial^2 f}{\partial A\partial\sigma_{A,t}}d\langle A, \sigma_A\rangle_t\right] \\
&\quad + \left[\frac{E_t^J - E_t}{E_t}\right]dN_A(t)
\end{aligned}
\tag{9}
$$

where $\Delta_t = \partial f/\partial A_t$ is the "delta" in option pricing, $\nu_t = \partial f/\partial\sigma_{A,t}$ is the so-called "vega" of the option, and $\langle X\rangle_t$ denotes the quadratic variation process for an arbitrary stochastic process X_t. Additionally, E_t^J is the value of equity for an asset jump of $J_A = J$. Hence, E_t^J is itself a random variable. Here we have ignored the sensitivity of the option value to the maturity of the debt.[19] In our applications, τ will be large enough that this assumption is innocuous. All

---

[19]For simplicity, we also ignore sensitivity to the risk-free rate, which is trivially satisfied if we assume a constant term structure.

the quadratic variation terms are of the order $\mathcal{O}(dt)$ and henceforth we collapse them to an unspecified function $q(A_t, \sigma_{A,t}; f)$, where the notation captures the dependence of the higher order Itō terms on the partial derivatives of the call option pricing function.

In reality, we do not observe A_t because it is the market value of assets. However, given that the call option pricing function is monotonically increasing in its first argument, it is safe to assume that $f(\cdot)$ is invertible with respect to this argument. We further assume that the call pricing function is homogenous of degree one in its first two arguments, which is a standard assumption in the option pricing literature. Define the inverse call option formula as follows:

$$\frac{A_t}{D_t} = g\left(E_t/D_t, 1, \sigma_{A,t}, J_A, N_A, \tau_t, r_t; \Theta_p, \Theta_r\right)$$

$$\equiv f^{-1}\left(E_t/D_t, 1, \sigma_{A,t}, J_A, N_A, \tau_t, r_t; \Theta_p, \Theta_r\right) \tag{10}$$

Thus, Equation (9) reduces returns to the following:[20]

$$\frac{dE_t}{E_t} = \overbrace{\Delta_t \cdot g_t \cdot \frac{D_t}{E_t}}^{\equiv LM(\cdot)} \times \frac{dA_t}{A_t} + \frac{\nu_t}{E_t} \cdot d\sigma_{A,t} + q(A_t, \sigma_{A,t}^f; f)dt + \left[\frac{E_t^J - E_t}{E_t}\right] dN_A(t)$$

$$= LM\left(E_t/D_t, 1, \sigma_{A,t}, J_A, N_A, \tau_t, r_t; \Theta_p, \Theta_r\right) \times \frac{dA_t}{A_t} +$$

$$\frac{\nu_t}{E_t} \cdot d\sigma_{A,t} + q(A_t, \sigma_{A,t}^f; f)dt + \left[\frac{E_t^J - E_t}{E_t}\right] dN_A(t) \tag{11}$$

Henceforth, when it is obvious, we will drop the functional dependence of the leverage multiplier on leverage, etc. and instead denote it simply by LM_t. In order to obtain a complete law of motion for equity, we use Itō's Lemma to derive the volatility process:

$$d\sigma_{A,t} = \overbrace{\left[\frac{\mu_v(t, v_t)}{2\sigma_{A,t}} - \frac{\sigma_v^2(t, v_t)}{8\sigma_{A,t}^3}\right]}^{\equiv s\left(\sigma_{A,t}; \mu_v, \sigma_v\right)} dt + \frac{\sigma_v(t, v_t)}{2\sigma_{A,t}} dB_v(t)$$

$$= s\left(\sigma_{A,t}; \mu_v, \sigma_v\right) dt + \frac{\sigma_v(t, v_t)}{2\sigma_{A,t}} dB_v(t) \tag{12}$$

Plugging Equations (8), (12) into Equation (11) yields the desired full equation of motion for equity returns:

$$\frac{dE_t}{E_t} = \left[LM_t\mu_A(t) + s\left(\sigma_{A,t}; \mu_v, \sigma_v\right) + q(A_t, \sigma_{A,t}; f)\right] dt$$

$$+ LM_t\sigma_{A,t}dB_A(t) + \frac{\nu_t}{E_t}\frac{\sigma_v(t, \sigma_{A,t})}{2\sigma_{A,t}}dB_v(t) + \left[\frac{E_t^J - E_t}{E_t}\right] dN_A(t) \tag{13}$$

---

[20]Using the fact that $f(\cdot)$ is homogenous of degree 1 in its first argument also implies that the $\Delta(\cdot)$ is the same. So with an inverse option pricing formula, $g(\cdot)$ in hand we can define the delta in terms of inverse leverage E_t/D_t.

Since typical daily equity returns are virtually zero on average, we can ignore the equity drift term. Instantaneous equity returns then naturally derive from Equation (13) with no drift:

$$\frac{dE_t}{E_t} = LM_t \sigma_{A,t} dB_A(t) + \frac{\nu_t}{E_t} \frac{\sigma_v(t, \sigma_{A,t})}{2\sigma_{A,t}} dB_v(t) + \left[\frac{E_t^J - E_t}{E_t} \right] dN_A(t) \tag{14}$$

It is important to notice here that Equation (14) describes equity returns under the *physical* measure. However, this does not mean that risk-adjustments are not relevant in this setup. Rather, all of the pricing of risks are captured in the leverage multiplier and the option vega.

Our ultimate object of interest is the instantaneous volatility of equity, but in order to obtain a complete expression for equity volatility we have to determine the variance of the jump component of equity returns. In Section 2 of the Online Appendix, we derive an easily computed expression, denoted by $\mathcal{V}_A^J (J_A, N_A(t); A_t)$, that involves a simple integration over the normal density. Hence, total instantaneous equity volatility in this (reasonably) general setting is given by:

$$vol_t \left(\frac{dE_t}{E_t} \right) = \sqrt{LM_t^2 \times \sigma_{A,t}^2 + \frac{\nu_t^2 \sigma_v^2(t, \sigma_{A.t})}{4E_t^2 \sigma_{A,t}^2} + 2LM_t \sigma_{A,t} \frac{\nu_t}{E_t} \frac{\sigma_v(t, \sigma_{A,t})}{2\sigma_{A,t}} \rho_t + \mathcal{V}_A^J (J_A, N_A(t); A_t)}$$

$$\tag{15}$$

There are four terms that contribute to equity volatility. The first term relates to asset volatility and the second relates to the volatility of asset volatility (as well as the sensitivity of the option to changes in volatility). The third depends on the correlation between assets innovations and asset volatility innovations. In practice, this correlation is negative. Thus, the middle two terms in Equation (15) will have offsetting effects in terms of the contribution of stochastic asset volatility to equity volatility. Finally, the fourth term relates to the volatility of the jump process for assets. In the Online Appendix (Section 1), we show that we can ignore all but the first term for the purposes of volatility modeling in our context because, compared to asset volatility, they contribute very little to equity volatility. This argument obviously relies on the assumption that the volatility of the volatility and the volatility contribution of the jump process are not too large. In the Online Appendix, we confirm this to be the case for parameterizations that accord with the previous literature (e.g. Bakshi, Cao, and Chen (1997)).

In lieu of this discussion, Equation (14) and (15) reduce to a very simple expression for equity returns and instantaneous volatility, and is the basis for the workhorse equation that we use throughout the paper:

$$\begin{aligned} \frac{dE_t}{E_t} &\approx LM_t \sigma_{A,t} dB_A(t) \\ vol_t \left(\frac{dE_t}{E_t} \right) &\approx LM_t \times \sigma_{A,t} \end{aligned} \tag{16}$$

Equation (16) is our key relationship of interest. It states that equity volatility (returns) is a scaled function of asset volatility (returns), where the function depends on financial leverage, D_t/E_t, as well as asset volatility over the life of the option (and the interest rate). The moniker of the "leverage multiplier" should be clear now. The functional form for $LM(\cdot)$ depends on a particular option pricing model, and one of our key contributions is to estimate a generalized

$LM(\cdot)$ function that, in practice, provides a good approximation for a number of option pricing models.

A.2   The Leverage Multiplier in the Black-Scholes-Merton Model

Computing the leverage multiplier in the Black-Scholes-Merton model is a straightforward application of the general formulation presented in Appendix A.1:

$$LM^{BSM}(D_t/E_t, \sigma_{A,t}, \tau_t, r_t) = \Delta_t^{BSM}(g_t^{BSM}, 1, \sigma_{A,t}, \tau_t, r_t) \times g_t^{BSM}(E_t/D_t, 1, \sigma_{A,t}, \tau_t, r_t) \cdot \frac{D_t}{E_t}$$

where g_t^{BSM} is the inverse BSM-call pricing function evaluated at a strike of 1 and a call price of E_t/D_t. In other words, it is the value of the underlying such that the BSM call option value would be E_t/D_t at a strike of 1. Δ_t^{BSM} is the "delta" of the BSM call option pricing formula, evaluated at g_t^{BSM} and a strike of 1.

To compute the BSM leverage multiplier in practice, we proceed in three steps:

1. Using the observed E_t/D_t ratio, invert the BSM call option formula to get an implied A_t/D_t ratio. To do this, set the strike equal to 1 (we have used the homogeneity of the BSM formula with respect to the underlying asset value). Another choice that needs to be made is what volatility to input into the inverse BSM formula. See Section INSERT for how we make this choice.

2. Using the BSM-implied A_t/D_t ratio, compute the BSM delta.

3. The BSM leverage multiplier is the BSM delta $\times$ the BSM implied A_t/D_t divided by the observed E_t/D_t ratio.

In our full recursive volatility model, the leverage multiplier is lagged, so these computations are done with lagged values.

B Appendix: Details on Generating the Leverage Multiplier In Different Option Pricing Settings

This appendix details how we compute the leverage multiplier in various option pricing settings. The end goal is to determine the relationship between the leverage multiplier and leverage, which we define as the ratio of the face value of debt to the market value of equity. In all three models we consider, we set the initial asset value $A_0 = 1$, the risk free rate $r_f = 3\%$, and the time to maturity of the debt $\tau = 2$ years. Computing the leverage multiplier then requires us to trace out call option values (i.e. equity values) and call option deltas (i.e. how equity values change with respect to A_0) over varying face values of debt D. Keep in mind that in all models, we are choosing the risk-neutral parameters that govern asset returns and asset volatility, though this distinction is not relevant for the Black-Scholes-Merton model.

The Leverage Multiplier in the Black-Scholes-Merton Model

We set $\sigma_A = 17.5\%$.

The Leverage Multiplier in the Bakshi, Cao, and Chen (1997) Model

In the Bakshi, Cao, and Chen (1997), risk-neutral asset returns evolve according to:

$$
\begin{aligned}
\frac{dA_t}{A_t} &= \left[r_{ft} - \lambda_J \mu_J \right] dt + \sigma_{A,t} dB_A(t) + J(t) dq(t) \\
d\sigma_{A,t}^2 &= \left[\theta_v - \kappa_v \sigma_{A,t}^2 \right] dt + \sigma_v \sigma_{A,t} dB_v(t)
\end{aligned}
$$

where $q(t)$ is a Poisson counting process with constant intensity λ_J. $J(t)$ is the percentage jump size and is i.i.d lognormal with unconditional mean μ_J and standard deviation σ_J. The parameter θ_v dictates the long-run average of risk-neutral asset volatility and κ_v determines the speed of mean reversion. Finally, the correlation between $dB_v(t)$ and $dB_A(t)$ is given by ρ.

We set $\lambda_J = 0.4, \mu_J = -0.1, \sigma_J = 0.15$. This corresponds to 0.4 jumps a year, each with an average jump size of -10% and volatility of 15%. Additionally, we set $\kappa_v = 2.77$ and $\theta_v = 0.013$. This means that the half-life of volatility is about 3 months (roughly what is typically found in the literature, e.g. Christoffersen, Heston, and Jacobs (2013)) and the unconditional volatility of asset returns is 17.5%. We also set $\sigma_v = 17.5\%$ and $\rho = -0.7$. When computing the leverage multiplier, we set the current spot volatility equal to its long run average, so $\sigma_{A,0} = 17.5\%$.

The Leverage Multiplier in a GARCH Model

When pricing options on discrete-time GARCH processes, there is often no closed form solution for call prices, necessitating the use of simulation techniques. Thus, to compute the leverage multiplier in this context, we simulate risk-neutral asset returns from an initial asset value of $A_0 = 1$. The simulation generates a set of terminal values, A_T, which in turn generate an equity value for a given face value of debt D. Specifically, $(1 + r_f)^\tau \times E(D) = \frac{1}{S} \sum_{s=1}^{S} \max(A_{T,s}, D)$, where $A_{T,s}$ is the simulated terminal asset value in simulation s, and S is the total number of simulations. We then compute numerical derivatives to measure how

the equity value changes with respect to A_0. Finally, we obtain the leverage multiplier over varying D/E values.

The asymmetric GARCH model that we use for risk-neutral asset returns is as follows:

$$
\begin{aligned}
r_{A,t} &= \sqrt{h_{A,t}}\varepsilon_{A,t} \\
h_{A,t} &= \omega + \alpha r_{A,t-1}^2 + \gamma r_{A,t-1}^2 1_{r_{A,t-1}<0} + \beta h_{A,t-1}
\end{aligned}
$$

where $1_{x<a}$ is an indicator variable that equals 1 if $x < a$ and 0 otherwise. $\varepsilon_{A,t}$ is a standard normal shock. The parameter γ captures volatility asymmetry and a positive γ effectively captures the negative correlation between asset returns and volatility. We parameterize the risk-neutral GARCH process as: $\omega = 4.86e - 7$, $\alpha = 0.022$, $\gamma = 0.18$, $\beta = 0.884$. These parameters imply that the long-run risk-neutral volatility of assets is also 17.5%. We choose a large γ to emphasize the impact that volatility asymmetry has on the leverage multiplier.

C Appendix: Cumulative Asset Volatility Forecast in the Structural GARCH Model

Recall that asset variance in the Structural GARCH model evolves according to:

$$h_{A,t+1} = \omega + \alpha r_{A,t-1}^2 + \gamma \alpha r_{A,t-1}^2 1_{r_{A,t-1}<0} + \beta h_{A,t}$$

We want to know the cumulative volatility forecast for assets over the period $t+1, ..., t+h$, where h is the number of days until maturity. In the notation of the model, this is simply $\tau_t \times 365$. Some tedious algebra implies the following closed form for this forecast:

$$\sum_{i=1}^{h} h_{A,t+i} = \frac{\omega}{1-\theta}h - \frac{\omega}{1-\theta} \cdot \frac{1-\theta^h}{1-\theta} + \frac{1-\theta^h}{1-\theta} \times h_{A,t+1}$$

where $\theta = \alpha + \gamma/2 + \beta$. Note that $h_{A,t+1}$ is in the information set at time t.

D Appendix: Measuring Realized Variance

Depending on data availability, we compute a daily measure of realized variance using 5-minute intraday returns. This means we just sum up the squared 5-minute intraday returns within each day. We generally follow the steps suggested by Andersen, Bollerslev, Diebold, and Labys (2003) in cleaning the intraday TAQ data, though the cleaning choices are not as big of an issues given that we are using 5-min returns and not tick-by-tick data. The main steps in this process are as follows: (i) if several trades have the same time stamp, we use the median price; (ii) we only use trades from one exchange for each asset; (iii) we filter out trades that are flagged in any way by TAQ; (iv) 5 minute returns that are larger than 10% in absolute value are set to zero, though these are very rare; (v) if the realized variance for a given 5-minute subsample is larger than 3 times the median RV of all the subsamples in a given day, that subsample is ignored.

We were able to compute a 5-minute realized variance series for 42 out of the 91 firms in our sample. For each firm and date pair, we count the number of trades that occurred for that firm. We further discard firm-date observations where the number of trades is in the bottom 5% of our entire sample. This mainly affects firms at the early part of the sample given that higher frequency trading was still in its early stages.

E   Appendix: Precautionary Capital

To compute PCAP, we simulate a bivariate process for the firm's equity return, denoted $r_{i,t}^E$, and the market's equity return, denoted $r_{m,t}^E$. The bivariate process we adopt is described as follows:

$$
\begin{aligned}
r_{m,t}^E &= \sqrt{h_{m,t}^E}\,\varepsilon_{m,t} \\
r_{i,t}^E &= \sqrt{h_{i,t}^E}\,\varepsilon_{i,t} \\
&\quad \sqrt{h_{i,t}^E}\left(\rho_{i,t}\varepsilon_{m,t} + \sqrt{1-\rho_{i,t}^2}\,\xi_{i,t}\right) \\
(\varepsilon_{m,t},\xi_{i,t}) &\sim F
\end{aligned}
$$

where the shocks $(\varepsilon_{m,t},\xi_{i,t})$ are serially-independent and identically distributed over time and have zero mean, unit variance, and zero covariance. We do not assume the two shocks are independent, however, and allow them to have extreme tail dependence nonparametrically.[21] The processes $h_{m,t}^E$, $h_{i,t}^E$ and $\rho_{i,t}$ represent the conditional variance of the market, the conditional variance of the firm, and the conditional correlation between the market and the firm, respectively. It is important to note that under the Structural GARCH model, we are really estimating correlations between shocks to the equity market index and shocks to firm asset returns.

  Simulation proceeds in a few steps:

1. Estimate a GJR(1,1) model for the market ($h_{m,t}^E$)

2. Estimate either a GJR(1,1) model or Structural GARCH model to the firm's equity ($h_{i,t}^E$)

3. Using the standardized residuals from Step 1 and 2, estimate a bivariate DCC(1,1) model (Engle (2002)).

4. Filter the idiosyncratic firm shocks ξ_{it} using the estimated time-series of correlations from the DCC:
$$
\xi_{it} = \frac{\varepsilon_{it} - \rho_{i,t}\varepsilon_{m,t}}{\sqrt{1-\rho_{i,t}^2}}
$$

5. To simulate, we bootstrap the market shocks $\varepsilon_{m,t}$ and the idiosyncratic firm shocks ξ_{it}. We do so by randomly drawing days from the estimation sample and taking the shocks ε_{mt} and ξ_{it} from that day. When bootstrapping we use the same set of random days regardless of the volatility model that we use for the firm's equity (i.e. Structural GARCH versus GJR).

6. Simulate the market volatility process forward using the bootstrapped shocks ε_{mt}. Save paths where the market drops 10 percent over the next month. These are "crisis paths".

7. Using the DCC parameters, the DCC recursion, and bootstrapped ξ_{it} along crisis paths, generate a series of simulated firm standardized residuals ε_{it}.

---

[21]See Brownlees and Engle (Forthcoming) for complete details.

8. Using the simulated firm standardized residuals, simulate the firm's equity returns using the desired volatility process. We declare any path where leverage (D/E) exceeds 150 as a bankruptcy path.

This delivers us a series of terminal equity returns for the firm, which we can combine with the firm's initial equity value to compute terminal equity values. We then count the percent of paths that exceed our desired terminal equity value (e.g. the capital requirement). When using the Structural GARCH model, we repeat this process for different starting values of equity. This allows us to trace out a function mapping initial equity values to likelihoods of meeting the capital requirement. We use a bisection search algorithm to find the initial equity value that delivers our desired confidence level, and PCAP follows naturally from this value. When using the GJR model for the firm, we do not need to search over initial equity values, since terminal equity returns are invariant to initial capital structure. In this case, we compute the quantiles of the terminal equity return and adjust the initial equity until we achieve the desired level of confidence for meeting the capital requirement. Efficient simulation of a bivariate Structural GARCH process is, however, not trivial. The MATLAB© code for this purpose is available from the authors upon request.

F Appendix: Full List of Firms for Main Analysis

AEL*	BOKF	FGL	IVZ	PNC	UNM*
AFG*	BRK (A)*	FHN	JPM	PRU	USB
AFL*	C	FITB	KEY	RF	V
AIG	CBSH	FMER	L	RGA*	VLY
AIZ*	CFR	FNB	LNC	RJF	VOYA*
ALL	CIT	FNFG	MET	SBNY*	WBS
AMP	CMA	GNW	MKL*	SCHW*	WFC
AMTD	CME	GS	MS	SIVB	WRB*
ANAT*	CNA	HBAN	MTB	SNV	WTFC
ASB	CNO	HBHC	NNI	STI	Y*
AXP	COF*	HIG	NTRS*	STT	ZION
BAC	DFS	HRG*	NYCB*	TCB	
BBT	ETFC	IBKC	PACW	TCBI	
BK*	EVER	IBKR	PB	TMK	
BKU	EWBC	ICE*	PBCT*	TRV	
BLK*	FCNCA	ISBC	PFG	UMPQ	

Notes: * indicates the firm had a estimated $\phi = 0$ in the Structural GARCH model